

# Non-Gaussian Observations in Nonlinear Compressed Sensing via Stein Discrepancies

Larry Goldstein<sup>\*,†</sup>, and Xiaohan Wei<sup>†</sup>  
 e-mail: larry@usc.edu; xiaohanw@usc.edu

**Abstract:** Performance guarantees for compression in nonlinear models under non-Gaussian observations can be achieved through the use of distributional characteristics that are sensitive to the distance to normality, and which in particular return the value of zero under Gaussian or linear sensing. The use of these characteristics, or discrepancies, improves some previous results in this area by relaxing conditions and tightening performance bounds. In addition, these characteristics are tractable to compute when Gaussian sensing is corrupted by either additive errors or mixing.

## 1. Introduction

Consider the following nonlinear sensing model, where  $(y_1, \mathbf{a}_1), \dots, (y_m, \mathbf{a}_m)$  in  $\mathbb{R} \times \mathbb{R}^d$  are i.i.d. copies of an observation and sensing vector pair  $(y, \mathbf{a})$  satisfying

$$E[y|\mathbf{a}] = \theta(\langle \mathbf{a}, \mathbf{x} \rangle), \quad (1)$$

where  $\mathbf{a}$  is composed of entry-wise independent random variables distributed as  $a$ , a mean zero, variance one random variable. Throughout the paper we further assume that the function  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  is measurable, and  $\mathbf{x} \in \mathbb{R}^d$  is an unknown, non-zero vector lying in a set  $K \subseteq \mathbb{R}^d$ . The goal is to recover  $\mathbf{x}$  given the measurement pairs  $\{(y_i, \mathbf{a}_i)\}_{i=1}^m$ . We note that the magnitude of  $\mathbf{x}$  is unidentifiable under the model (1) as  $\theta(\cdot)$  is unknown. Hence in the following we assume  $\|\mathbf{x}\|_2 = 1$  without loss of generality.

In [ALPV14], the authors consider model (1) under the one-bit sensing scenario where  $y_1, \dots, y_m$  lie in  $\{-1, 1\}$  and  $\theta : \mathbb{R} \rightarrow [-1, 1]$ . They demonstrate that despite  $\theta$  being unknown and potentially highly non-linear, performance guarantees can be provided for estimators  $\hat{\mathbf{x}}$  of  $\mathbf{x}$  without additional knowledge of the structure of  $\theta$ , and in a way that allows for non-Gaussian sensing.

Consideration of the non-Gaussian case introduces some challenges, reflected in potentially poor performance of the bounds, additional smoothness assumptions, and difficulties that may arise when the unknown is extremely sparse. We show these difficulties can be overcome in many regards through the introduction of various measures of the discrepancy between the sensing distribution of  $a$  and standard normal  $g$ . Though our main goal is to develop bounds that are sensitive to certain deviations from normality, and which in particular recover the previous results for Gaussian sensing and linear sensing as special cases, we also provide explicit small constants in our recovery bounds.

### 1.1. Estimator and main result

Given the pairs  $\{(y_i, \mathbf{a}_i)\}_{i=1}^m$  let

$$L_m(\mathbf{t}) := \|\mathbf{t}\|_2^2 - \frac{2}{m} \sum_{i=1}^m y_i \langle \mathbf{a}_i, \mathbf{t} \rangle \quad \text{for } \mathbf{t} \in K, \quad (2)$$

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<sup>\*</sup>Department of Mathematics, University of Southern California

<sup>†</sup>Department of Electrical Engineering, University of Southern California

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which is an unbiased estimator of

$$L(\mathbf{t}) := \|\mathbf{t}\|_2^2 - 2E[y \langle \mathbf{a}, \mathbf{t} \rangle]. \quad (3)$$

As  $E[\mathbf{a}\mathbf{a}^T] = \mathbf{I}_{d \times d}$ , minimizing  $L(\mathbf{t})$  is equivalent to minimizing the quadratic loss  $E[(y - \langle \mathbf{a}, \mathbf{t} \rangle)^2]$ . Thus, we define the estimator

$$\hat{\mathbf{x}}_m := \operatorname{argmin}_{\mathbf{t} \in K} L_m(\mathbf{t}). \quad (4)$$

For simplicity of notation, we will write

$$f_{\mathbf{x}}(\mathbf{t}) := \frac{1}{m} \sum_{i=1}^m y_i \langle \mathbf{a}_i, \mathbf{t} \rangle. \quad (5)$$

To state the main result, we need the following three definitions:

**Definition 1.1** (Gaussian mean width). *For  $\mathbf{g} \sim \mathcal{N}(0, \mathbf{I}_{d \times d})$ , the Gaussian mean width of a set  $\mathcal{T} \subseteq \mathbb{R}^d$  is*

$$\omega(\mathcal{T}) = E \left[ \sup_{\mathbf{t} \in \mathcal{T}} \langle \mathbf{g}, \mathbf{t} \rangle \right].$$

**Remark 1.1.** In [ALPV14], the definition of Gaussian mean width of a set  $\mathcal{T}$  is taken to be

$$\omega(\mathcal{T}) = E \left[ \sup_{\mathbf{t} \in \mathcal{T} - \mathcal{T}} \langle \mathbf{g}, \mathbf{t} \rangle \right],$$

where the supremum is over the symmetric difference of the set  $\mathcal{T}$ . Here for the ease of presentation, we adopt the somewhat more “classical” Definition 1.1 that appears in earlier works in the literature, such as [RV08]. These two definitions are indeed equivalent up to constants because for any fixed  $\mathcal{T}$ ,  $E[\sup_{\mathbf{t} \in \mathcal{T}} \langle \mathbf{g}, \mathbf{t} \rangle] \leq E[\sup_{\mathbf{t} \in \mathcal{T} - \mathcal{T}} \langle \mathbf{g}, \mathbf{t} \rangle] \leq 2E[\sup_{\mathbf{t} \in \mathcal{T}} \langle \mathbf{g}, \mathbf{t} \rangle]$ .

**Remark 1.2** (Measurability issue). *The precise meaning of  $E[\sup_{t \in \mathcal{T}} X(t)]$  for an arbitrary process  $\{X(t)\}_{t \in \mathcal{T}}$  is not clear if  $\mathcal{T}$  is uncountable. In fact, for an uncountable index set  $\mathcal{T}$ , the function  $\sup_{t \in \mathcal{T}} X(t)$  might not be measurable. Well known counter examples exist even under the case where the function  $X(\cdot)$  is jointly measurable on the product space  $(\Omega \times \mathcal{T}, \mathcal{E} \otimes \Psi)$  (first constructed by Luzin and Suslin), where  $\mathcal{E}$  and  $\Psi$  are Borel  $\sigma$ -algebras on  $\Omega$  and  $\mathcal{T}$  respectively. However, when  $\mathcal{T}$  is a measurable subset of  $\mathbb{R}^d$  (which is the case we are interested in) and  $X(\cdot)$  is jointly measurable on  $(\Omega \times \mathcal{T}, \mathcal{E} \otimes \Psi)$ , one can show that the  $\sup_{t \in \mathcal{T}} X(t)$  is always measurable.*

*Indeed,  $\sup_{t \in \mathcal{T}} X(t)$  is measurable if and only if the set  $\{\sup_{t \in \mathcal{T}} X(t) > c\} \in \mathcal{E}$  for any  $c \in \mathbb{R}$ . On the other hand,  $\{\sup_{t \in \mathcal{T}} X(t) > c\} = P_\Omega\{X(\cdot) > c\}$ , where for any set  $A \in \Omega \times \mathcal{T}$ ,  $P_\Omega A := \{\omega \in \Omega : (\omega, t) \in A\}$  is the projection of the set  $A$  onto  $\Omega$ . Then, the measurability comes from the following theorem in [Co80]: If  $(\Omega, \mathcal{E})$  is a measurable space and  $\mathcal{T}$  is a Polish space, then, the projection onto  $\Omega$  of any product measurable subset of  $\Omega \times \mathcal{T}$  is also measurable.*

**Definition 1.2** ( $\psi_q$ -norm). *The  $\psi_q$ -norm of a real valued random variable  $X$  is given by*

$$\|X\|_{\psi_q} = \sup_{p \geq 1} p^{-\frac{1}{q}} (E[|X|^p])^{\frac{1}{p}}.$$

*In particular, for  $q = 1$  and  $q = 2$  respectively, the value of  $\psi_q$  is called the subexponential and subgaussian norm, and we say  $X$  is subexponential or subgaussian when  $\|X\|_{\psi_1} < \infty$  or  $\|X\|_{\psi_2} < \infty$ .*

**Remark 1.3.** It is easily justified that  $\|\cdot\|_{\psi_q}$  for  $q \geq 1$  defines a norm with identification of almost everywhere equal random variables. Here we only check the triangle inequality as it is immediate that  $\|\cdot\|_{\psi_q}$  is homogeneous and separates points. Indeed, for any two random variables  $X$  and  $Y$ , the Minkowski inequality yields that

$$\|X + Y\|_{\psi_q} = \sup_{p \geq 1} p^{-\frac{1}{q}} (E[|X + Y|^p])^{\frac{1}{p}} \leq \sup_{p \geq 1} p^{-\frac{1}{q}} \left( (E[|X|^p])^{\frac{1}{p}} + (E[|Y|^p])^{\frac{1}{p}} \right) \leq \|X\|_{\psi_q} + \|Y\|_{\psi_q}.$$

**Definition 1.3** (Descent cone). The descent cone of a set  $\mathcal{T} \subseteq \mathbb{R}^d$  at any point  $\mathbf{t}_0 \in \mathcal{T}$  is defined as

$$D(\mathcal{T}, \mathbf{t}_0) = \{\tau \mathbf{h} : \tau \geq 0, \mathbf{h} \in \mathcal{T} - \mathbf{t}_0\}.$$

In the following, we say a random variable  $a$  is symmetric if the distributions of  $a$  and  $-a$  are equal.

**Theorem 1.1.** Let  $\mathbf{a} = (a_1, \dots, a_d)$  where  $a_1, \dots, a_d$  are i.i.d. copies of a random variable  $a$  with symmetric subgaussian distribution having unit variance, and let  $\{(y_i, \mathbf{a}_i)\}_{i=1}^m$  be i.i.d. copies of the pair  $(y, \mathbf{a})$  where  $y$ , given by the sensing model (1), is assumed to be subgaussian. If  $K$  is a closed, measurable subset of  $\mathbb{R}^d$  and  $\lambda \mathbf{x} \in K$  for the scaling factor

$$\lambda = E[y \langle \mathbf{a}, \mathbf{x} \rangle], \quad (6)$$

then for all  $u \geq 2$ , with probability at least  $1 - 4e^{-u}$  the estimator  $\hat{\mathbf{x}}_m$  given by (4) satisfies

$$\|\hat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2 \leq 2\alpha + C_0(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(D(K, \lambda \mathbf{x}) \cap \mathbb{S}^{d-1}) + u}{\sqrt{m}},$$

for all  $m \geq \omega(D(K, \mathbf{x}) \cap \mathbb{S}^{d-1})^2$  and some constant  $C_0 > 0$ , where

$$\alpha = \sup\{|E[y \langle \mathbf{a}, \mathbf{t} \rangle] - \lambda \langle \mathbf{x}, \mathbf{t} \rangle|, \mathbf{t} \in B_2^d\}, \quad (7)$$

and  $\mathbb{S}^{d-1}$  and  $B_2^d$  are the unit Euclidean sphere and ball in  $\mathbb{R}^d$ , respectively.

**Remark 1.4.** In Lemma 2.2 of [ALPV14], a similar bound is presented under the additional assumptions that  $\{y_i\}_{i=1}^m$  take values in  $\{-1, 1\}$ ,  $\theta : \mathbb{R} \rightarrow [-1, 1]$  and that  $K$  lies in a unit Euclidean ball  $B_2^d$ . Specifically, under the preceeding assumptions it is shown that

$$\|\hat{\mathbf{x}}_m - \mathbf{x}\|_2^2 \leq \frac{4\alpha}{\lambda} + C\|a\|_{\psi_2} \frac{\omega(K) + \beta}{\lambda \sqrt{m}},$$

with probability at least  $1 - 4e^{-\beta^2}$ , where  $\lambda := E[g\theta(g)]$  and  $g \sim \mathcal{N}(0, 1)$ . Here, we are able to obtain a more general result at the extra cost of a term that is of the same order as previously existing ones in the bound, and in particular which vanish as  $m \rightarrow \infty$ .

Finally, since we assume  $y$  is subgaussian instead of just taking values in  $\{-1, 1\}$ , for any  $\mathbf{t} \in \mathbb{R}^d$ ,  $y \langle \mathbf{a}, \mathbf{t} \rangle$  is a sub-exponential random variable as oppose to sub-gaussian in [ALPV14]. This necessitates a generic chaining argument to obtain a sub-exponential concentration bound.

## 2. Discrepancy bounds via Stein's method

Here we introduce some measures of the sensing distribution's proximity to normality that can be used to bound  $\alpha$  in (7). Our measures yield the value zero when  $a$  is normal or the function  $\theta$  is linear, and hence Theorem 1.1 recovers results for the normal and linear compressed sensing

models as special cases. Additionally, we further improve upon the results of [ALPV14] with explicit small constants. Moreover, for the cases where the Gaussian sensing vector is corrupted by an additive error, or by mixing, in an amount measured by some  $\epsilon \in [0, 1]$ , one can supply simple explicit bounds that tend to zero in  $\epsilon$ , see Theorems 2.2 and 2.4.

In Sections 2.1 and 2.2 we consider the cases where  $\theta$  is a Lipschitz function, and the sign function, respectively; the difference in the degree of smoothness in these two cases necessitates the use of different ways of measuring the discrepancy to normality. In Section 2.3 we discuss the relationship of these measures to each other, and also to the total variation distance.

An observation that will be useful in both settings is that by definition (5), for any  $\mathbf{t} \in \mathbb{R}^d$ , we have

$$E[f_{\mathbf{x}}(\mathbf{t})] = E[y\langle \mathbf{a}, \mathbf{t} \rangle] = E[E[y\langle \mathbf{a}, \mathbf{t} \rangle | \mathbf{a}]] = E[\langle \mathbf{a}, \mathbf{t} \rangle \theta(\langle \mathbf{a}, \mathbf{x} \rangle)] = \langle \mathbf{v}_{\mathbf{x}}, \mathbf{t} \rangle$$

where  $\mathbf{v}_{\mathbf{x}} = E[\mathbf{a}\theta(\langle \mathbf{a}, \mathbf{x} \rangle)]$ , (8)

and we note by (6) that therefore

$$\lambda = \langle \mathbf{v}_{\mathbf{x}}, \mathbf{x} \rangle = E[\langle \mathbf{a}, \mathbf{x} \rangle \theta(\langle \mathbf{a}, \mathbf{x} \rangle)]. \quad (9)$$

In the settings of both Sections 2.1 and 2.2, we require facts regarding the zero bias distribution, and depend on [GR97] or [CGS10] for properties stated below. With  $\mathcal{L}(\cdot)$  denoting distribution, or law, given a mean zero distribution  $\mathcal{L}(a)$  with finite, non-zero variance  $\sigma^2$ , there exists a unique law  $\mathcal{L}(a^*)$ , termed the ‘ $a$ -zero bias’ distribution, characterized by the satisfaction of

$$E[af(a)] = \sigma^2 E[f'(a^*)] \quad \text{for all Lipschitz functions } f.$$

The existence of the variance of  $a$ , and hence also its second moment, guarantees that the expectation on the left, and hence also on the right, exists.

Letting

$$\text{Lip}_1 = \{g : \mathbb{R} \rightarrow \mathbb{R} \text{ satisfying } |g(y) - g(x)| \leq |y - x|\},$$

we recall that the Wasserstein, or  $L^1$  distance between the laws  $\mathcal{L}(X)$  and  $\mathcal{L}(Y)$  of two random variables  $X$  and  $Y$  can be defined as

$$d_1(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{f \in \text{Lip}_1} |Ef(X) - Ef(Y)|,$$

or alternatively as

$$d_1(\mathcal{L}(X), \mathcal{L}(Y)) = \inf_{(X, Y)} E|X - Y| \quad (10)$$

where the infimum is over all couplings  $(X, Y)$  of random variables having the given marginals. The infimum is achievable for real valued random variables, see [Ra91].

Now we may define

$$\gamma_{\mathcal{L}(a)} = d_1(a, a^*). \quad (11)$$

Stein’s characterization [St72] of the normal yields that  $\mathcal{L}(a^*) = \mathcal{L}(a)$  if and only if  $a$  is a mean zero normal variable. Further, with some abuse of notation, writing  $\gamma_a$  for (11) for simplicity, Lemma 1.1 of [G10] yields that if  $a$  has mean zero, variance 1 and finite third moment, then

$$\gamma_a \leq \frac{1}{2} E|a|^3, \quad (12)$$

so in particular  $\gamma_a < \infty$  whenever  $a$  has a finite third moment. In the case where  $Y_1, \dots, Y_n$  are independent mean zero random variables with finite, non-zero variances  $\sigma_1^2, \dots, \sigma_n^2$  and sum  $Y = \sum_{i=1}^n Y_i$  with variance  $\sigma^2$ , we may construct  $Y^*$  with the  $Y$ -zero biased distribution by letting

$$Y^* = Y - Y_I + Y_I^* \quad \text{where} \quad P[I = i] = \frac{\sigma_i^2}{\sigma^2}, \quad (13)$$

where  $Y_i^*$  has the  $Y_i$ -zero biased distribution and is independent of  $Y_j, j \neq i$ , and where the random index  $I$  is independent of  $\{Y_i, Y_i^*, i = 1, \dots, n\}$ . We will also make use of the fact that for any  $c \neq 0$

$$\mathcal{L}((ca)^*) = \mathcal{L}(ca^*). \quad (14)$$

### 2.1. Lipschitz functions

When  $\theta$  is a Lipschitz function not assumed to possess any additional smoothness, (18) of Theorem 2.1 below gives a bound on  $\alpha$  in (7) in terms of Stein coefficients. We say  $T$  is a Stein coefficient, or Stein kernel, for a random variable  $X$  with finite, non zero variance when

$$E[Xf(X)] = E[Tf'(X)] \quad (15)$$

for all Lipschitz functions  $f$ . Specializing (15) to the cases where  $f(x) = 1$  and  $f(x) = x$  we find

$$E[X] = 0 \quad \text{and} \quad \text{Var}(X) = E[T]. \quad (16)$$

By Stein's characterization [St72], the distribution of  $X$  is normal with mean zero and variance  $\sigma^2$  if and only if  $T = \sigma^2$ . If  $c$  is a non-zero constant and  $T_X$  is a Stein coefficient for  $X$ , then  $c^2 T_X$  is a Stein coefficient for  $Y = cX$ . Indeed, by setting  $g(x) = f(cx)$  we obtain

$$\begin{aligned} E[Yf(Y)] &= cE[Xf(cX)] = cE[Xg(X)] = cE[T_X g'(X)] \\ &= cE[cT_X f'(cX)] = E[c^2 T_X f'(Y)]. \end{aligned} \quad (17)$$

Stein coefficients first appeared in the work of [CP92], and were further developed in [Ch09] for random variables that are functions of Gaussians.

The following result considers two separate sets of hypotheses on the unknown function  $\theta$  and the sensing distribution  $a$ . The assumptions leading to the bound (18) require fewer conditions on  $\theta$  and more on  $a$  as compared to those leading to (19). We note that by Stein's characterization, when  $a$  is standard normal we may take  $T = 1$  in (18), and  $\gamma_a = 0$  in (19), and hence  $\alpha = 0$  in both the cases considered in the theorem that follows. The bound (19) also returns zero discrepancy in the special case where  $\theta$  is linear, and thus recovers the results on linear compressed sensing [RV08] when combined with Theorem 1.1.

For a real valued function  $f$  with domain  $D$  let

$$\|f\| = \sup_{x \in D} |f(x)|.$$

**Theorem 2.1.** *Let  $a$  be a mean zero, variance one random variable and set  $\mathbf{a} = (a_1, \dots, a_d)$  with  $a_1, \dots, a_d$  independent random variables distributed as  $a$ , and let  $\alpha$  be as in (7).*

(a) *If  $\theta \in \text{Lip}_1$  and  $a$  has Stein coefficient  $T$ , then*

$$\alpha \leq E|1 - T|. \quad (18)$$

(b) *If  $\theta$  possesses a bounded second derivative, then*

$$\alpha \leq \|\theta''\| \gamma_a. \quad (19)$$

**Remark 2.1.** In [ALPV14] the quantity  $\alpha$  is bounded in terms of the total variation distance  $d_{\text{TV}}(a, g)$  between  $a$  and the standard Gaussian distribution  $g$ . In particular, for  $\theta \in C^2$ , Proposition 5.5 of [ALPV14] yields

$$\alpha \leq 8(Ea^6 + Eg^6)^{1/2}(\|\theta'\| + \|\theta''\|)\sqrt{d_{\text{TV}}(a, g)}. \quad (20)$$

In contrast, the upper bound (18) does not depend on any moments of  $a$ , requires  $\theta$  to be only once differentiable, and in typical cases where  $d_{\text{TV}}(a, g)$  and  $E|1 - T|$  are of the same order, that is, when the upper bound in Lemma 2.5 is of the correct order,  $\alpha$  in (18) is bounded by a first power rather than the larger square root in (20).

When  $\theta$  possesses a bounded second derivative, the upper bound (19) improves on (20) in terms of constant factors, requirements on the existence of moments, and dependence on a first power rather than a square root. In this case Lemma 2.6 shows  $d_{\text{TV}}(a, g)$  and  $\gamma_a$  are of the same order when  $a$  has bounded support.

Measuring discrepancy from normality in terms of  $E|1 - T|$  and  $\gamma_a$  also has the advantage of additional tractability in the cases where each component of the Gaussian sensing vector  $\mathbf{g}$  has been independently corrupted at the level of some  $\epsilon \in [0, 1]$  by a non Gaussian, mean zero, variance one distribution  $a$ . In the two models we consider we let the sensing vector have i.i.d. entries, and hence only specify the distribution of its components. The first model is the case of additive error, where each component of the sensing vector is of the form

$$g_\epsilon = \sqrt{1 - \epsilon}g + \sqrt{\epsilon}a \quad (21)$$

with  $a$  independent of  $g$ , and the mixture model where each component has been corrupted due to some ‘bad event’  $A$  that substitutes  $g$  with  $a$  so that

$$g_\epsilon = g\mathbf{1}_{A^c} + a\mathbf{1}_A, \quad (22)$$

where  $A$  occurs with probability  $\epsilon$ , independently of  $g, a$  and a given Stein coefficient  $T$  for  $a$ . Since

$$E[Tf'(a)] = E[E[T|a]f'(a)] \quad (23)$$

we see that  $E[T|a]$  is a Stein coefficient for  $a$ . Hence, upon replacing  $T$  by  $E[T|a]$  only the independence of  $A$  from  $\{g, a\}$  is required.

Theorem 2.2 shows that under both scenarios (a) and (b) considered in Theorem 2.1, and further, under both the additive and mixture models, the value  $\alpha$  can be bounded explicitly in terms of a quantity that vanishes in  $\epsilon$ . Further, we note that both error models agree with each other, and with the model of Theorem 2.1, when  $\epsilon = 1$ , and that Theorem 2.2 recovers Theorem 2.1 when so specializing.

**Theorem 2.2.** Under condition (a) of Theorem 2.1, under both the additive (21) and mixture (22) error models, we have

$$\alpha \leq \epsilon E|1 - T|.$$

Under the additive error model (21),

$$\gamma_{g_\epsilon} \leq \epsilon^{3/2}\gamma_a, \quad \text{and when condition (b) of Theorem 2.1 holds,} \quad \alpha \leq \epsilon^{3/2}\|\theta''\|\gamma_a, \quad (24)$$

and under the mixture error model (22),

$$\gamma_{g_\epsilon} \leq \epsilon\gamma_a, \quad \text{and when condition (b) of Theorem 2.1 holds,} \quad \alpha \leq \epsilon\|\theta''\|\gamma_a. \quad (25)$$

*Proof:* By the assumptions of independence and those on the distributions, in both error models  $g_\epsilon$  has mean zero and variance 1. As the components of the sensing vector are i.i.d. by construction, the hypotheses on  $\mathbf{a}$  in Theorem 2.1 holds.

First consider scenario (a) and let the additive error model hold. If a random variable  $W$  is the sum of two independent mean zero variables  $X$  and  $Y$  with finite variances and Stein coefficients  $T_X$  and  $T_Y$  respectively, then for any Lipschitz function one has

$$\begin{aligned} E[Wf(W)] &= E[(X+Y)f(X+Y)] = E[Xf(X+Y)] + E[Yf(X+Y)] \\ &= E[T_X f'(X+Y)] + E[T_Y f'(X+Y)] = E[(T_X + T_Y)f'(X+Y)] \\ &= E[T_W f'(W)] \quad \text{where } T_W = T_X + T_Y, \end{aligned}$$

showing that Stein coefficients are additive for independent summands. In particular, now also using (17), we see that the Stein coefficient  $T_\epsilon$  for  $g_\epsilon$  in (21) is given by  $T_\epsilon = 1 - \epsilon + \epsilon T$ , where  $T$  is the Stein coefficient for  $a$ . As  $1 - T_\epsilon = \epsilon(1 - T)$ , the first claim of the lemma follows by applying Theorem 2.1.

For the mixture model, by the independence between  $A$  and  $\{a, g, T\}$ ,

$$\begin{aligned} E[g_\epsilon f(g_\epsilon)] &= (1 - \epsilon)E[gf(g)] + \epsilon E[af(a)] = (1 - \epsilon)E[f'(g)] + \epsilon E[Tf'(a)] \\ &= E[\mathbf{1}_{A^c} f'(g) + T \mathbf{1}_A f'(a)] = E[\mathbf{1}_{A^c} f'(g_\epsilon) + T \mathbf{1}_A f'(g_\epsilon)] = E[T_\epsilon f'(g_\epsilon)] \quad \text{where } T_\epsilon = \mathbf{1}_{A^c} + T \mathbf{1}_A. \end{aligned}$$

The second claim of the lemma now follows by applying Theorem 2.1, observing that  $1 - T_\epsilon = \mathbf{1}_A(1 - T)$  and applying the independence between  $T$  and  $A$ .

Now let scenario (b) hold, and again consider the additive error model. In this case, as the standard normal is a fixed point of the zero bias transformation, we may set  $g^* = g$ . In addition, we construct  $a^*$  to have the  $a$ -zero bias distribution, be independent of  $g$  and  $A$ , and achieve the infimum  $d_1(\mathcal{L}(a), \mathcal{L}(a^*))$  in (10). We claim that

$$g_\epsilon^* = (\sqrt{1 - \epsilon}g^* + \sqrt{\epsilon}a)\mathbf{1}_{A^c} + (\sqrt{1 - \epsilon}g + \sqrt{\epsilon}a^*)\mathbf{1}_A = \sqrt{1 - \epsilon}g + \sqrt{\epsilon}(a\mathbf{1}_{A^c} + a^*\mathbf{1}_A),$$

has the  $g_\epsilon$ -zero bias distribution. Indeed, (13) says one may construct the zero bias distribution of a sum of independent terms by choosing one proportional to variance and replacing it by a variable independent of the remaining summands and having the chosen summand's zero bias distribution, where the replacement is done independent of all else. Applying (14), yielding  $(\epsilon a)^* =_d \epsilon a^*$  and likewise  $(\sqrt{1 - \epsilon}g)^* =_d \sqrt{1 - \epsilon}g^*$ , shows the conditions to apply (13) are fulfilled. We now obtain

$$g_\epsilon^* - g_\epsilon = \sqrt{1 - \epsilon}g + \sqrt{\epsilon}(a\mathbf{1}_{A^c} + a^*\mathbf{1}_A) - (\sqrt{1 - \epsilon}g + \sqrt{\epsilon}a) = \sqrt{\epsilon}(a^* - a)\mathbf{1}_A.$$

As the Wasserstein distance is the infimum (10) over all couplings between  $g$  and  $g_\epsilon$ , and using that  $A$  is independent of  $a$  and  $a^*$ , we have

$$\gamma_{g_\epsilon} = d_1(g_\epsilon, g_\epsilon^*) \leq E|g_\epsilon^* - g_\epsilon| = \sqrt{\epsilon}E|a^* - a|P(A) = \epsilon^{3/2}\gamma_a.$$

The proof of the first claim under (b) can now be completed by applying (19).

Continuing under scenario (b), again consider the mixture model (22). By Theorem 2.1 of [G10], as  $\text{Var}(a) = \text{Var}(g)$ , the variable

$$g_\epsilon^* = g^*\mathbf{1}_{A^c} + a^*\mathbf{1}_A = g\mathbf{1}_{A^c} + a^*\mathbf{1}_A$$

has the  $g_\epsilon$  zero bias distribution, where we again take  $g^*$  and  $a^*$  as in the previous construction. Hence, arguing as for the additive error model, we obtain the bound

$$\gamma_{g_\epsilon} \leq E|g_\epsilon^* - g_\epsilon| = E[|a^* - a|\mathbf{1}_A] = \epsilon\gamma_a.$$



The second claim under (b) now follows as the first.  $\square$

*Proof of Theorem 2.1.* Recalling that  $\mathbf{x}$  is a unit vector, for any  $\mathbf{t} \in B_2^d$  the vectors  $\mathbf{x}$  and  $\mathbf{v} = \mathbf{t} - \langle \mathbf{x}, \mathbf{t} \rangle \mathbf{x}$  are perpendicular. If  $\mathbf{v} \neq 0$  set  $\mathbf{x}^\perp$  to be the unit vector in direction  $\mathbf{v}$ , and let  $\mathbf{x}^\perp$  be zero otherwise. These vectors produce the orthogonal decomposition of  $\mathbf{t} \in B_2^d$  as

$$\mathbf{t} = \langle \mathbf{x}, \mathbf{t} \rangle \mathbf{x} + \langle \mathbf{x}^\perp, \mathbf{t} \rangle \mathbf{x}^\perp. \quad (26)$$

Defining

$$Y = \langle \mathbf{a}, \mathbf{x} \rangle \quad \text{and} \quad Y^\perp = \langle \mathbf{a}, \mathbf{x}^\perp \rangle,$$

using the decomposition (26) in (8), and the expression for  $\lambda$  in (9) yields

$$\begin{aligned} E[f_{\mathbf{x}}(\mathbf{t})] &= E[\langle \mathbf{a}, \mathbf{t} \rangle \theta(\langle \mathbf{a}, \mathbf{x} \rangle)] = \langle \mathbf{x}, \mathbf{t} \rangle E[\langle \mathbf{a}, \mathbf{x} \rangle \theta(\langle \mathbf{a}, \mathbf{x} \rangle)] + \langle \mathbf{x}^\perp, \mathbf{t} \rangle E[\langle \mathbf{a}, \mathbf{x}^\perp \rangle \theta(\langle \mathbf{a}, \mathbf{x} \rangle)] \\ &= \lambda \langle \mathbf{x}, \mathbf{t} \rangle + \langle \mathbf{x}^\perp, \mathbf{t} \rangle E[Y^\perp \theta(Y)]. \end{aligned}$$

As  $\|\mathbf{x}^\perp\|_2$  and  $\|\mathbf{t}\|_2$  are at most one, applying the Cauchy-Schwarz inequality we obtain

$$|E[f_{\mathbf{x}}(\mathbf{t})] - \lambda \langle \mathbf{x}, \mathbf{t} \rangle| \leq \left| E[Y^\perp \theta(Y)] \right|. \quad (27)$$

We determine a Stein coefficient for  $Y^\perp$  as follows. For  $T_i$  Stein coefficients for  $a_i$ , independent and identically distributed as the given  $T$  for all  $i = 1, \dots, d$ , by conditioning on  $Y - x_i a_i$ , a function of  $\{a_j, j \neq i\}$  and therefore independent of  $a_i$ , using (17) we have

$$\begin{aligned} E[x_i^\perp a_i \theta(Y)] &= E[x_i^\perp a_i \theta(x_i a_i + (Y - x_i a_i))] = E[x_i^\perp x_i T_i \theta'(x_i a_i + (Y - x_i a_i))] \\ &= E[x_i^\perp x_i T_i \theta'(Y)]. \end{aligned} \quad (28)$$

Hence

$$\begin{aligned} E[Y^\perp \theta(Y)] &= \sum_{i=1}^d E[x_i^\perp a_i \theta(Y)] = E[T_{Y^\perp} \theta'(Y)] \\ \text{where } T_{Y^\perp} &= \sum_{i=1}^d x_i^\perp x_i T_i = \sum_{i=1}^d x_i^\perp x_i (T_i - 1), \end{aligned} \quad (29)$$

where the last equality follows from  $\langle \mathbf{x}, \mathbf{x}^\perp \rangle = 0$ .

Now from (27) and (29) we have

$$\begin{aligned} |E[f_{\mathbf{x}}(\mathbf{t})] - \lambda \langle \mathbf{x}, \mathbf{t} \rangle| &\leq |E[T_{Y^\perp} \theta'(Y)]| \\ &\leq E|T_{Y^\perp}| \leq \sum_{i=1}^d |x_i^\perp x_i| E|T - 1| \leq \|\mathbf{x}^\perp\|_2 \|\mathbf{x}\|_2 E|T - 1| \leq E|T - 1|, \end{aligned}$$

using  $\theta \in \text{Lip}_1$  in the second inequality, followed by (29) again and the Cauchy-Schwarz inequality, noting that  $\|\mathbf{x}^\perp\|_2 \leq 1$  and  $\|\mathbf{x}\|_2 = 1$ . Hence we obtain

$$|E[f_{\mathbf{x}}(\mathbf{t})] - \lambda \langle \mathbf{x}, \mathbf{t} \rangle| \leq E|T - 1| \quad \text{for all } \mathbf{t} \in B_2^d,$$



which completes the proof of (18) in light of the definition (7) of  $\alpha$ .

In a similar fashion, if  $\theta$  is twice differentiable with bounded second derivative, then in place of (28), for every  $i = 1, \dots, d$  we may write

$$E[x_i^\perp a_i \theta(Y)] = E \left[ x_i^\perp a_i \theta(x_i a_i + (Y - x_i a_i)) \right] = E \left[ x_i^\perp x_i \theta'(x_i a_i^* + (Y - x_i a_i)) \right],$$

where  $a_i, a_i^*$  are constructed on the same space so that  $a_i^*$  has the  $a_i$ -zero biased distribution and that the infimum in (10) is achieved. Hence,

$$\begin{aligned} E[Y^\perp \theta(Y)] &= \sum_{i=1}^d E[x_i^\perp a_i \theta(Y)] = \sum_{i=1}^d E \left[ x_i^\perp x_i \theta'(x_i a_i^* + (Y - x_i a_i)) \right] \\ &= \sum_{i=1}^d E \left[ x_i^\perp x_i (\theta'(x_i a_i^* + (Y - x_i a_i)) - \theta'(Y)) \right] \\ &= \sum_{i=1}^d E \left[ x_i^\perp x_i (\theta'(x_i a_i^* + (Y - x_i a_i)) - \theta'(x_i a_i + (Y - x_i a_i))) \right], \quad (30) \end{aligned}$$

where, as in (29), we have used  $\langle \mathbf{x}^\perp, \mathbf{x} \rangle = 0$  for the third equality.

The proof of (19) is completed by applying (27) and (30) to obtain

$$\begin{aligned} |E[f_{\mathbf{x}}(\mathbf{t})] - \lambda \langle \mathbf{x}, \mathbf{t} \rangle| &\leq |E[Y^\perp \theta(Y)]| \leq \|\theta''\|_\infty \sum_{i=1}^d E \left| x_i^\perp x_i^2 (a_i^* - a_i) \right| \leq \|\theta''\|_\infty \gamma_a \sum_{i=1}^d |x_i^\perp x_i^2| \\ &\leq \|\theta''\|_\infty \gamma_a \sum_{i=1}^d |x_i^\perp x_i| \leq \|\theta''\|_\infty \gamma_a, \end{aligned}$$

where we have applied the mean value theorem for the second inequality, the fact that the infimum in (10) is achieved for the third, that  $\|\mathbf{x}\|_2 = 1$  for the fourth, and the Cauchy-Schwarz inequality for the last.  $\square$

## 2.2. Sign function

In this section we consider the case where  $\theta$  is the sign function given by

$$\theta(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

The motivation comes from the one bit compressed sensing model, see [ALPV14] for a more detailed discussion. The following result shows how  $\alpha$  of (7) can be bounded in terms of the discrepancy measure  $\gamma_a$  introduced in Section 2.1. Throughout this section set

$$c_1 = \sqrt{2/\pi} - 1/2.$$

We continue to assume that the unknown  $\mathbf{x}$  has unit Euclidean length.

**Theorem 2.3.** *With  $\gamma_a$  defined in (11), if  $\|\mathbf{x}\|_3^3 \leq c_1/\gamma_a$  and  $\|\mathbf{x}\|_\infty \leq 1/2$ , then  $\alpha$  defined in (7) satisfies*

$$\alpha \leq (10\gamma_a E|a|^3 \|\mathbf{x}\|_\infty)^{1/2}.$$

Under the condition that  $\|\mathbf{x}\|_\infty \leq c/E|a|^3$  for some  $c > 0$ , Proposition 4.1 of [ALPV14] yields the existence of a constant  $C$  such that

$$\alpha \leq CE|a|^3 \|\mathbf{x}\|_\infty^{1/2}. \quad (31)$$

Theorem 2.3 improves (31) by introducing the factor of  $\gamma_a$  in the bound, thus providing a right hand side that takes the value 0 when  $a$  is normal. Applying inequality (12) to the bound of Lemma 2.1 in the case where  $a$  has finite third moment recovers (31) with  $C$  assigned the specific value of  $\sqrt{5}$ .

In terms of the total variation distance between  $a$  and the Gaussian  $g$ , in Proposition 5.2 [ALPV14] provides the bound

$$\alpha \leq C(Ea^4)^{1/8} d_{\text{TV}}(a, g)^{1/8}$$

depending on an unspecified constant and an eighth root. For distributions where  $\gamma_a$  is comparable to the total variation distance, see Section 2.3, the bound of Theorem 2.3 would be preferred, and is also explicit.

Now we derive bounds on  $\alpha$  defined in (7) for the two error models introduced in Section 2.1. As there, the bounds vanish as  $\epsilon$  tends to zero. We note that Theorem 2.3 is recovered as the special case  $\epsilon = 1$  for both models considered.

**Theorem 2.4.** *In the additive and mixture error models (21) and (22), the bound of Theorem 2.3 becomes, respectively*

$$\alpha \leq \left( 10\epsilon^{3/2}\gamma_a \left( \sqrt{1-\epsilon} \left( \sqrt{\frac{8}{\pi}} \right)^{1/3} + \sqrt{\epsilon}E[|a|^3]^{1/3} \right)^3 \|\mathbf{x}\|_\infty \right)^{1/2}$$

and

$$\alpha \leq \left( 10\epsilon\gamma_a \left( (1-\epsilon) \left( \sqrt{\frac{8}{\pi}} \right)^{1/3} + \epsilon E[|a|^3]^{1/3} \right)^3 \|\mathbf{x}\|_\infty \right)^{1/2}.$$

We first demonstrate the proof of Theorem 2.3, starting with a series of lemmas.

**Lemma 2.1.** *For any mean zero, variance 1 random variable  $a$ , and any  $\mathbf{x} \in B_2^d$ ,*

$$\left| \langle \mathbf{v}_{\mathbf{x}}, \mathbf{x} \rangle - \sqrt{\frac{2}{\pi}} \right| \leq \gamma_a \|\mathbf{x}\|_3^3, \quad (32)$$

where  $\mathbf{v}_{\mathbf{x}} = E[\mathbf{a}\theta(\langle \mathbf{a}, \mathbf{x} \rangle)]$  as in (8).

The inequality in Lemma 2.1 should be compared to Lemma 5.3 of [ALPV14], where the bound on the quantity in (32) is in terms of the fourth root of the total variation distance between  $a$  and  $g$  and their fourth moments.

*Proof:* It is direct to verify that  $E|g| = \sqrt{2/\pi}$  for  $g \sim \mathcal{N}(0, 1)$ . In Lemma B.1 in Appendix B, we show that when taking  $f$  to be the unique bounded solution to the Stein equation

$$f'(x) - xf(x) = |x| - \sqrt{\frac{2}{\pi}}, \quad (33)$$

we have  $\|f''\|_\infty = 1$ , where  $\|\cdot\|_\infty$  is the essential supremum. Hence for a mean zero, variance one random variable  $Y$ , using that the sets of measure zero do not affect the integral below, we have

$$\begin{aligned} |E|Y| - E|g|| &= |E[f'(Y) - Yf(Y)]| = |E[f'(Y) - f'(Y^*)]| = \left| E \left[ \int_Y^{Y^*} f''(u) du \right] \right| \\ &\leq \|f''\|_\infty E|Y^* - Y| = E|Y^* - Y|, \end{aligned}$$

where  $Y^*$  is any random variable on the same space as  $Y$ , having the  $Y$ -zero biased distribution.

As  $\theta$  is the sign function

$$\langle \mathbf{v}_\mathbf{x}, \mathbf{x} \rangle = E[\langle \mathbf{a}, \mathbf{x} \rangle \theta(\langle \mathbf{a}, \mathbf{x} \rangle)] = E|\langle \mathbf{a}, \mathbf{x} \rangle| \quad \text{and hence} \quad \left| \langle \mathbf{v}_\mathbf{x}, \mathbf{x} \rangle - \sqrt{\frac{2}{\pi}} \right| = |E|\langle \mathbf{a}, \mathbf{x} \rangle| - E|g||.$$

For the case at hand, let  $Y = \langle \mathbf{a}, \mathbf{x} \rangle = \sum_{i=1}^n x_i a_i$ , where  $a_1, \dots, a_n$  are independent and identically distributed as  $a$ , having mean zero and variance 1 and recall  $\|\mathbf{x}\|_2 = 1$ . Then with  $P[I = i] = x_i^2 / \|\mathbf{x}\|_2$ , taking  $(a_i, a_i^*)$  to achieve the infimum in (10), that is, so that  $E|a_i^* - a_i| = d_1(a_i, a_i^*)$ , by (13) we obtain

$$E|Y^* - Y| = E|x_I(a_I^* - a_I)| = \sum_{i=1}^n \frac{|x_i|^3}{\|\mathbf{x}\|_2} \gamma_{a_i} = \gamma_a \|\mathbf{x}\|_3^3,$$

as desired.  $\square$

We now provide a version of Lemma 4.4 of [ALPV14] in terms of  $\gamma_a$  and specific constants.

**Lemma 2.2.** *The vector  $\mathbf{v}_\mathbf{x}$  in (8) satisfies  $\|\mathbf{v}_\mathbf{x}\|_2 \leq 1$ , and if  $\|\mathbf{x}\|_3^3 \leq c_1 / \gamma_a$  then*

$$\frac{1}{2} \leq \|\mathbf{v}_\mathbf{x}\|_2.$$

*Proof:* The upper bound follows as in the proof Lemma 4.4 in [ALPV14]. Slightly modifying the lower bound argument there through the use of Lemma 2.1 for the second inequality below we obtain

$$\|\mathbf{v}_\mathbf{x}\|_2 = \|\mathbf{v}_\mathbf{x}\|_2 \|\mathbf{x}\|_2 \geq |\langle \mathbf{v}_\mathbf{x}, \mathbf{x} \rangle| \geq \sqrt{\frac{2}{\pi}} - \gamma_a \|\mathbf{x}\|_3^3 \geq \sqrt{\frac{2}{\pi}} - c_1 = 1/2.$$

$\square$

We provide the following version of Lemma 4.5 of [ALPV14] with the explicit constant 2, and include a symmetry assumption on  $a$  that was used there implicitly.

**Lemma 2.3.** *If  $\|\mathbf{x}\|_\infty \leq 1/2$  and  $a$  has a symmetric distribution then the vector  $\mathbf{v}_\mathbf{x}$  in (8) satisfies*

$$\|\mathbf{v}_\mathbf{x}\|_\infty \leq 2E|a|^3 \|\mathbf{x}\|_\infty.$$

*Proof:* We follow the proof of Lemma 4.5 in [ALPV14], and for a given coordinate index  $i$  let  $S^{(i)} = \langle \mathbf{a}, \mathbf{x} \rangle - a_i x_i$ ; by the symmetry of  $a$  we can take  $x_i \geq 0$  without loss of generality. Using symmetry again in the second equality below and setting  $\tau_i^2 = \sum_{k \neq i} x_k^2$ , for fixed  $r$  we obtain

$$\begin{aligned} |E\theta(S^{(i)} + rx_i)| &= |P[S^{(i)} \geq -rx_i] - P[S^{(i)} < -rx_i]| \\ &= P[|S^{(i)}| \leq rx_i] = P[|S^{(i)}|/\tau_i \leq rx_i/\tau_i] \\ &\leq P[|g| \leq rx_i/\tau_i] + |P[|S^{(i)}|/\tau_i \leq rx_i/\tau_i] - P[|g| \leq rx_i/\tau_i]|. \end{aligned}$$

The hypothesis  $\|\mathbf{x}\|_\infty \leq 1/2$  implies  $\tau_i^2 \geq 3/4$ . Hence, using the supremum bound on the standard normal density for the first term and that  $\sqrt{8/3\pi} \leq 1$ , and the Berry-Esseen bound of [Sh10] with constant 0.56 on the second term, noting  $0.56(4/3)^{3/2} \leq 1$ , and that  $\|\mathbf{x}\|_3^3 \leq \|\mathbf{x}\|_\infty$  since  $\|\mathbf{x}\|_2 = 1$ , we obtain

$$|E[r\theta(S^{(i)} + rx_i)]| \leq r^2 x_i + |r| \|\mathbf{x}\|_\infty E|a|^3.$$

Considering now the  $i^{\text{th}}$  coordinate of  $\mathbf{v}_\mathbf{x} = E[\mathbf{a}\theta(\langle \mathbf{a}, \mathbf{x} \rangle)]$ , using  $E|a| \leq (Ea^2)^{1/2} = 1 \leq (E|a|^3)^{1/3} \leq E|a|^3$ , we have

$$|E[a_i\theta(\langle \mathbf{a}, \mathbf{x} \rangle)]| = |E[a_i\theta(S^{(i)} + x_i a_i)]| \leq x_i + \|\mathbf{x}\|_\infty E|a|^3 \leq 2E|a|^3 \|\mathbf{x}\|_\infty.$$

□

*Proof of Theorem 2.3:* We follow the proof of Proposition 4.1 of [ALPV14]. As  $\langle \mathbf{x}, \mathbf{v}_\mathbf{x} \rangle \neq 0$  we conclude  $\mathbf{v}_\mathbf{x} \neq 0$ , and defining  $\mathbf{z} = \mathbf{v}_\mathbf{x} / \|\mathbf{v}_\mathbf{x}\|_2$ , from Lemmas 2.2 and 2.3

$$\|\mathbf{z}\|_\infty = \frac{\|\mathbf{v}_\mathbf{x}\|_\infty}{\|\mathbf{v}_\mathbf{x}\|_2} \leq 2\|\mathbf{v}_\mathbf{x}\|_\infty \leq 4E|a|^3 \|\mathbf{x}\|_\infty.$$

Hence, with the second inequality from Lemma 2.1,

$$\begin{aligned} \|\mathbf{v}_\mathbf{x}\|_2 = \langle \mathbf{v}_\mathbf{x}, \mathbf{z} \rangle &= E[\theta(\langle \mathbf{a}, \mathbf{x} \rangle) \langle \mathbf{a}, \mathbf{z} \rangle] \leq E[\theta(\langle \mathbf{a}, \mathbf{z} \rangle) \langle \mathbf{a}, \mathbf{z} \rangle] \leq \sqrt{\frac{2}{\pi}} + \gamma_a \|\mathbf{z}\|_\infty \\ &\leq \sqrt{\frac{2}{\pi}} + 4\gamma_a E|a|^3 \|\mathbf{x}\|_\infty. \end{aligned} \quad (34)$$

Next, using (8), we bound  $|E[f_\mathbf{x}(\mathbf{t})] - \lambda \langle \mathbf{x}, \mathbf{t} \rangle| = |\langle \mathbf{v}_\mathbf{x}, \mathbf{t} \rangle - \lambda \langle \mathbf{x}, \mathbf{t} \rangle|$ . By the Cauchy-Schwartz inequality, now taking  $\mathbf{t} \in B_2^d$ ,

$$|\langle \mathbf{v}_\mathbf{x}, \mathbf{t} \rangle - \lambda \langle \mathbf{x}, \mathbf{t} \rangle|^2 = |\langle \mathbf{v}_\mathbf{x} - \lambda \mathbf{x}, \mathbf{t} \rangle|^2 \leq \|\mathbf{v}_\mathbf{x} - \lambda \mathbf{x}\|^2.$$

Furthermore, by (9), we have  $\langle \mathbf{v}_\mathbf{x}, \mathbf{x} \rangle = \lambda$ , thus

$$\begin{aligned} \|\mathbf{v}_\mathbf{x} - \lambda \mathbf{x}\|^2 &= \|\mathbf{v}_\mathbf{x}\|_2^2 - \lambda^2 + 2\lambda(\lambda - \langle \mathbf{v}_\mathbf{x}, \mathbf{x} \rangle) = (\|\mathbf{v}_\mathbf{x}\|_2 - \lambda)(\|\mathbf{v}_\mathbf{x}\|_2 + \lambda) \leq 2(\|\mathbf{v}_\mathbf{x}\|_2 - \lambda) \\ &= 2 \left( \|\mathbf{v}_\mathbf{x}\|_2 - \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}} - \lambda \right) \leq 10\gamma_a E|a|^3 \|\mathbf{x}\|_\infty, \end{aligned}$$

where we have applied Lemma 2.2 in the first inequality and the last inequality follows from (34), Lemma 2.1 and that  $E|a|^3 \geq 1$ . Now taking a square root finishes the proof. □

*Proof of Theorem 2.4:* Under the additive error model (21), by Minkowski's inequality

$$\begin{aligned} E[|g_\epsilon|^3]^{1/3} &= E[|\sqrt{1-\epsilon}g + \sqrt{\epsilon}a|^3]^{1/3} \leq \sqrt{1-\epsilon}E[|g|^3]^{1/3} + \sqrt{\epsilon}E[|a|^3]^{1/3} \\ &= \sqrt{1-\epsilon} \left( \sqrt{\frac{8}{\pi}} \right)^{1/3} + \sqrt{\epsilon}E[|a|^3]^{1/3}. \end{aligned}$$

Using this inequality and (24) in Theorem 2.3 gives the discrepancy bound in the additive error case.

For the mixture model (22), again by Minkowski's inequality,

$$\begin{aligned} E[|g_\epsilon|^3]^{1/3} &= E[|g\mathbf{1}_{A^c} + a\mathbf{1}_A|^3]^{1/3} \leq (1 - \epsilon)E[|g|^3]^{1/3} + \epsilon E[|a|^3]^{1/3} \\ &= (1 - \epsilon) \left( \sqrt{\frac{8}{\pi}} \right)^{1/3} + \epsilon E[|a|^3]^{1/3}. \end{aligned}$$

Using this inequality and (25) in Theorem 2.3 gives the discrepancy bound in the mixed error case.  $\square$

### 2.3. Relations between Measures of Discrepancy

So far we have considered two methods for handling non-Gaussian sensing, the first using Stein coefficients and the second by the zero bias distribution. In this section we discuss some relations between these two, and also their connections to the total variation distance  $d_{\text{TV}}(\cdot, \cdot)$  appearing in the bound of [ALPV14] and discussed in Remark 2.1.

The following result appears in Section 7 of [Go07].

**Lemma 2.4.** *If  $a$  is a mean zero, variance 1 random variable, and  $a^*$  has the  $a$ -zero biased distribution, then*

$$d_{\text{TV}}(a, g) \leq 2d_{\text{TV}}(a, a^*). \quad (35)$$

The following related result is from [Ch09].

**Lemma 2.5.** *If the mean zero, variance 1 random variable  $a$  has Stein coefficient  $T$ , then*

$$d_{\text{TV}}(a, g) \leq 2E|1 - T|,$$

where  $g \sim \mathcal{N}(0, 1)$ .

From (23), if  $T$  is a Stein coefficient for  $a$  then so is  $h(a) = E[T|a]$ . Comparing with the identity that characterizes the zero bias distribution  $a^*$ , we obtain

$$E[f'(a^*)] = E[af(a)] = E[h(a)f'(a)].$$

In particular, when such a  $T$  exists, the distribution of  $a^*$  is absolutely continuous with respect to that of  $a$ , with Radon Nikodym derivative  $h(a)$ . When  $a$  is a mean zero, variance one random variable with density  $p(a)$  whose support is a possibly infinite interval, then using the form of density  $p^*(a)$  of  $a^*$  in [GR97], we obtain

$$p^*(y) = E[a\mathbf{1}(a > y)] \quad \text{and} \quad h(y) = \frac{p^*(y)}{p(y)}\mathbf{1}(p(y) > 0) = \frac{E[a\mathbf{1}(a > y)]}{p(y)}\mathbf{1}(p(y) > 0). \quad (36)$$

In particular, in this case

$$E|1 - h(a)| = \int_{y:p(y)>0} \left| 1 - \frac{p^*(y)}{p(y)} \right| p(y) dy = \int_{\mathbb{R}} |p(y) - p^*(y)| dy = d_{\text{TV}}(a, a^*),$$

and the bounds in Lemmas 2.5 and 2.4 are equal. Overall then, in the case where the Stein coefficient of a random variable is given as a function of the random variable itself, the discrepancy considered in part (a) of Theorem 2.2 is simply the total variation distance between  $a$  and  $a^*$ , while the discrepancies in part (b) of Theorem 2.2 and Section 2.2 are the Wasserstein distance.

To provide a concrete example of a Stein coefficient, a simple computation using (36) shows that if  $a$  has the double exponential distribution with variance 1 and density

$$p(y) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|y|} \quad \text{then} \quad h(y) = \frac{1}{2}(1 + \sqrt{2}|y|).$$

In this case

$$E|1 - h(a)| = E|1 - \sqrt{2}a|\mathbf{1}(a > 0) = \frac{1}{e}.$$

We note that this discrepancy value is smaller than the square root of the total variation, a value larger than 0.531, that appears in the bound of [ALPV14] of Remark 2.1.

The following result, providing a bound complementary to (35), taken together with Lemma 2.4 shows that  $d_{\text{TV}}(a, a^*)$  and  $d_{\text{TV}}(a, g)$  are of the same order in general for distributions of bounded support.

**Lemma 2.6.** *Let  $a$  be a mean zero random, variance one random variable with density  $p(y)$  supported in  $[-b, b]$ . Then*

$$d_{\text{TV}}(a, a^*) \leq (1 + b^2)d_{\text{TV}}(a, g).$$

**Proof:** With  $p^*(y)$  the density of  $a^*$  given by (36), we have

$$d_{\text{TV}}(a, a^*) = \int_{[-b, b]} |p(y) - p^*(y)| dy = \int_{[-b, b]} (p(y) - p^*(y)) \phi(y) dy = E\phi(a) - E\phi(a^*),$$

where

$$\phi(y) = \begin{cases} 1 & p(y) \geq p^*(y) \\ -1 & p(y) < p^*(y). \end{cases}$$

Setting

$$f(y) = \int_0^y \phi(u) du \quad \text{and} \quad h(y) = \phi(y) - y \int_0^y \phi(u) du,$$

we have  $f'(y) = \phi(y)$ , and

$$d_{\text{TV}}(a, a^*) = E[f'(a) - f'(a^*)] = E[f'(a) - af(a)] = Eh(a) - Eh(g).$$

For  $y \in [-b, b]$  we have  $|h(y)| \leq |\phi(y)| + |y| \int_0^y |\phi(u)| du \leq 1 + b^2$ , hence

$$d_{\text{TV}}(a, a^*) \leq (1 + b^2)d_{\text{TV}}(a, g),$$

as claimed.  $\square$

We make a brief mention of the fact that Stein coefficients can be constructed in some generality due to a result of [Ch09]. In particular,

$$h(\mathbf{g}) = \int_0^\infty e^{-t} \langle \nabla F(\mathbf{g}), \widehat{E}(\nabla F(\mathbf{g}_t)) \rangle dt$$

is a Stein coefficient for a function  $F(\mathbf{g})$  of a standard normal vector in  $\mathbb{R}^n$  when  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable and  $\mathbf{g}_t = e^{-t}\mathbf{g} + \sqrt{1 - e^{-2t}}\widehat{\mathbf{g}}$ , where  $\widehat{\mathbf{g}}$  is an independent copy of  $\mathbf{g}$  and  $\widehat{E}$  integrates over  $\widehat{\mathbf{g}}$ , that is, takes conditional expectation with respect to  $\mathbf{g}$ .

### 3. Proof of Theorem 1.1

So far, we have shown that the penalty  $\alpha$  for non-normality in (7) of Theorem 1.1 can be bounded explicitly using discrepancy measures that arise in Stein's method. In this section, we focus on proving Theorem 1.1 via a generic chaining argument.

#### 3.1. Preliminaries

The following notions are necessary ingredients in the generic chaining argument. Let  $(\mathcal{T}, d)$  be a metric space. If  $\mathcal{A}_l \subseteq \mathcal{A}_{l+1} \subseteq \mathcal{T}$  for every  $l \geq 0$  we say  $\{\mathcal{A}_l\}_{l=0}^\infty$  is an increasing sequence of subsets of  $\mathcal{T}$ . Let  $N_0 = 1$  and  $N_l = 2^{2^l}$ ,  $\forall l \geq 1$ .

**Definition 3.1** (Admissible sequence). *An increasing sequence of subsets  $\{\mathcal{A}_l\}_{l=0}^\infty$  of  $\mathcal{T}$  is admissible if  $|\mathcal{A}_l| \leq N_l$  for all  $l \geq 0$ .*

Essentially following the framework of Section 2.2 of [Ta14], for each subset  $\mathcal{A}_l$ , we define  $\pi_l : \mathcal{T} \rightarrow \mathcal{A}_l$  as the closest point map  $\pi_l(\mathbf{t}) = \arg \min_{\mathbf{s} \in \mathcal{A}_l} d(\mathbf{s}, \mathbf{t})$ ,  $\forall \mathbf{t} \in \mathcal{T}$ . Since each  $\mathcal{A}_l$  is a finite set, the minimum is always achievable. If the argmin is not unique a representative is chosen arbitrarily. The Talagrand  $\gamma_2$ -functional is defined as

$$\gamma_2(\mathcal{T}, d) := \inf \sup_{\mathbf{t} \in \mathcal{T}} \sum_{l=0}^{\infty} 2^{l/2} d(\mathbf{t}, \pi_l(\mathbf{t})), \quad (37)$$

where the infimum is taken with respect to all admissible sequences. Though there is no guarantee that  $\gamma_2(\mathcal{T}, d)$  is finite, the following majorizing measure theorem tells us that its value is comparable to the supremum of a certain Gaussian process.

**Lemma 3.1** (Theorem 2.4.1 of [Ta14]). *Consider a family of centered Gaussian random variables  $\{G(\mathbf{t})\}_{\mathbf{t} \in \mathcal{T}}$  indexed by  $\mathcal{T}$ , with the canonical distance*

$$d(\mathbf{s}, \mathbf{t}) = E[(G(\mathbf{s}) - G(\mathbf{t}))^2]^{1/2}, \quad \forall \mathbf{s}, \mathbf{t} \in \mathcal{T}.$$

*Then for a universal constant  $L$  that does not depend on the covariance of the Gaussian family, we have*

$$\frac{1}{L} \gamma_2(\mathcal{T}, d) \leq E \left[ \sup_{\mathbf{t} \in \mathcal{T}} G(\mathbf{t}) \right] \leq L \gamma_2(\mathcal{T}, d).$$

In addition, we need the following notions and propositions; we recall the  $\psi_q$  norms from Definition 1.2.

**Definition 3.2** (Subgaussian random vector). *A random vector  $\mathbf{X} \in \mathbb{R}^d$  is subgaussian if the random variables  $\langle \mathbf{X}, \mathbf{z} \rangle$ ,  $\mathbf{z} \in \mathbb{S}^{d-1}$  are subgaussian with uniformly bounded subgaussian norm. The corresponding subgaussian norm of the vector  $\mathbf{X}$  is then given by*

$$\|\mathbf{X}\|_{\psi_2} = \sup_{\mathbf{z} \in \mathbb{S}^{d-1}} \|\langle \mathbf{X}, \mathbf{z} \rangle\|_{\psi_2}.$$

The proof of the following three propositions are shown in the Appendix.

**Proposition 3.1.** *If both  $X$  and  $Y$  are subgaussian random variables, then  $XY$  is an subexponential random variable satisfying*

$$\|XY\|_{\psi_1} \leq 2\|X\|_{\psi_2}\|Y\|_{\psi_2}.$$



**Proposition 3.2.** *If  $\mathbf{a}$  is a subgaussian random vector with covariance matrix  $\Sigma$ , then*

$$\sigma_{\max}(\Sigma) \leq 2\|\mathbf{a}\|_{\psi_2}^2,$$

where  $\sigma_{\max}(\cdot)$  denotes the maximal singular value of a matrix.

**Proposition 3.3.** *Consider a random vector  $\mathbf{a} \in \mathbb{R}^d$ , where each entry  $a_i$  is an i.i.d. copy of a symmetric subgaussian random variable  $a$ . Then,  $\mathbf{a}$  is a subgaussian random vector with norm  $\|\mathbf{a}\|_{\psi_2} \leq C\|a\|_{\psi_2}$  where  $C$  is a positive constant independent of the dimension  $d$ .*

### 3.2. Proof Structure

By (2), (4) and (5),

$$\hat{\mathbf{x}}_m = \operatorname{argmin}_{\mathbf{t} \in K} (\|\mathbf{t}\|_2^2 - 2f_{\mathbf{x}}(\mathbf{t})).$$

In order to demonstrate that  $\hat{\mathbf{x}}_m$  is a good estimate of  $\lambda\mathbf{x}$ , we need to control the mean of  $f_{\mathbf{x}}(\cdot)$  and the deviation of  $f_{\mathbf{x}}(\cdot)$  from its mean. As shown in the previous section, the mean of  $f_{\mathbf{x}}(\cdot)$  can be effectively characterized through the introduced discrepancy measures. The deviation is controlled by the following lemma:

**Lemma 3.2** (Concentration). *Under the assumptions of Theorem 1.1, for all  $u \geq 2$  and  $m \geq \omega(D(K, \lambda\mathbf{x}) \cap \mathbb{S}^{d-1})^2$ ,*

$$P \left[ \sup_{\mathbf{t} \in D(K, \lambda\mathbf{x}) \cap \mathbb{S}^{d-1}} |f_{\mathbf{x}}(\mathbf{t}) - E[f_{\mathbf{x}}(\mathbf{t})]| \geq C_0(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(D(K, \lambda\mathbf{x}) \cap \mathbb{S}^{d-1}) + u}{\sqrt{m}} \right] \leq 4e^{-u},$$

where  $C_0 > 0$  is a fixed constant<sup>1</sup>.

The proof, which is provided in the next subsection, is based on the improved chaining technique introduced in [Di13]. We now show that once Lemma 3.2 is proved how Theorem 1.1 follows without much overhead.

**Lemma 3.3.** *For any  $\mathbf{z} \in K$ , we have*

$$L(\mathbf{z}) - L(\lambda\mathbf{x}) \geq \|\mathbf{z} - \lambda\mathbf{x}\|_2^2 - 2\alpha\|\mathbf{z} - \lambda\mathbf{x}\|_2,$$

where  $L(\cdot)$ ,  $\lambda$  and  $\alpha$  are defined in (3), (6) and (7), respectively.

*Proof.* For any  $\mathbf{z} \in K$ , recalling  $E[f_{\mathbf{x}}(\mathbf{t})] = E[y \langle \mathbf{a}, \mathbf{t} \rangle]$ ,

$$\begin{aligned} L(\mathbf{z}) - L(\lambda\mathbf{x}) &= \|\mathbf{z}\|_2^2 - \|\lambda\mathbf{x}\|_2^2 - 2E[y \langle \mathbf{a}, \mathbf{z} - \lambda\mathbf{x} \rangle] = \|\mathbf{z}\|_2^2 - \|\lambda\mathbf{x}\|_2^2 - 2E[f_{\mathbf{x}}(\mathbf{z} - \lambda\mathbf{x})] \\ &\geq \|\mathbf{z}\|_2^2 - \|\lambda\mathbf{x}\|_2^2 - 2\lambda \langle \mathbf{z} - \lambda\mathbf{x}, \mathbf{x} \rangle - 2\alpha\|\mathbf{z} - \lambda\mathbf{x}\|_2 \\ &= \|\mathbf{z} - \lambda\mathbf{x}\|_2^2 - 2\alpha\|\mathbf{z} - \lambda\mathbf{x}\|_2, \end{aligned}$$

where the inequality follows from  $|E[f_{\mathbf{x}}(\mathbf{t})] - \lambda \langle \mathbf{x}, \mathbf{t} \rangle| \leq \alpha\|\mathbf{t}\|_2$  for any  $\mathbf{t} \in \mathbb{R}^d$ , a consequence of (7).  $\square$

---

<sup>1</sup>By taking  $\mathcal{T} = D(K, \lambda\mathbf{x}) \cap \mathbb{S}^{d-1} \subseteq \mathbb{R}^d$  in the Remark 1.2, we have that the supremum is indeed measurable in the probability space  $(\Omega, \mathcal{E}, P)$ .

By Lemma 3.3, we have

$$\begin{aligned}
\|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2^2 &\leq L(\widehat{\mathbf{x}}_m) - L(\lambda \mathbf{x}) + 2\alpha \|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2 \\
&= L(\widehat{\mathbf{x}}_m) - L_m(\widehat{\mathbf{x}}_m) + L_m(\widehat{\mathbf{x}}_m) - L_m(\lambda \mathbf{x}) + L_m(\lambda \mathbf{x}) - L(\lambda \mathbf{x}) + 2\alpha \|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2 \\
&= -2(E_m[y \langle \mathbf{a}, \widehat{\mathbf{x}}_m \rangle] - f_{\mathbf{x}}(\widehat{\mathbf{x}}_m)) + L_m(\widehat{\mathbf{x}}_m) - L_m(\lambda \mathbf{x}) + 2(E_m[y \langle \mathbf{a}, \lambda \mathbf{x} \rangle] - f_{\mathbf{x}}(\lambda \mathbf{x})) \\
&\quad + 2\alpha \|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2 \\
&\leq 2|f_{\mathbf{x}}(\widehat{\mathbf{x}}_m - \lambda \mathbf{x}) - E_m[y \langle \mathbf{a}, \widehat{\mathbf{x}}_m - \lambda \mathbf{x} \rangle]| + L_m(\widehat{\mathbf{x}}_m) - L_m(\lambda \mathbf{x}) + 2\alpha \|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2,
\end{aligned}$$

where  $E_m[\cdot]$  the conditional expectation given  $\{(\mathbf{a}_i, y_i)\}_{i=1}^m$ . Since  $\widehat{\mathbf{x}}_m$  solves (4) and  $\lambda \mathbf{x} \in K$ , it follows that  $L_m(\widehat{\mathbf{x}}_m) - L_m(\lambda \mathbf{x}) \leq 0$ . Thus,

$$\|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2^2 \leq 2|f_{\mathbf{x}}(\widehat{\mathbf{x}}_m - \lambda \mathbf{x}) - E_m[y \langle \mathbf{a}, \widehat{\mathbf{x}}_m - \lambda \mathbf{x} \rangle]| + 2\alpha \|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2.$$

Since  $\widehat{\mathbf{x}}_m - \lambda \mathbf{x} \in D(K, \lambda \mathbf{x})$ , dividing both sides by  $\|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2$  and using the fact that for any fixed  $\mathbf{t} \in \mathbb{R}^d$ ,  $E[f_{\mathbf{x}}(\mathbf{t})] = E[y \langle \mathbf{a}, \mathbf{t} \rangle]$  gives

$$\|\widehat{\mathbf{x}}_m - \lambda \mathbf{x}\|_2 \leq 2 \sup_{\mathbf{t} \in D(K, \lambda \mathbf{x}) \cap \mathbb{S}^{d-1}} |f_{\mathbf{x}}(\mathbf{t}) - E[f_{\mathbf{x}}(\mathbf{t})]| + 2\alpha.$$

Now applying Lemma 3.2 finishes the proof of Theorem 1.1.

### 3.3. Proving Lemma 3.2 via Generic Chaining

For  $\mathcal{T} \subseteq \mathbb{R}^d$  and  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$  we write  $\gamma_2(\mathcal{T})$  to denote  $\gamma_2(\mathcal{T}, \|\cdot\|_2)$  defined in (37). Defining the Gaussian process  $G(\mathbf{t}) = \langle \mathbf{g}, \mathbf{t} \rangle$ ,  $\mathbf{t} \in \mathcal{T}$ , with  $\mathbf{g} \sim \mathcal{N}(0, \mathbf{I}_{d \times d})$  we have

$$E[(G(t) - G(s))^2]^{1/2} = \|t - s\|_2, \quad \forall t, s \in \mathcal{T}.$$

When  $\mathcal{T}$  is bounded we may conclude that  $\omega(\mathcal{T}) < \infty$  directly from Definition 1.1, and Lemma 3.1 then implies that Gaussian mean width  $\omega(\mathcal{T})$  and  $\gamma_2(\mathcal{T})$  are of the same order, i.e. there exists a universal constant  $L \geq 1$  independent of  $\mathcal{T}$  such that

$$\frac{1}{L} \gamma_2(\mathcal{T}) \leq \omega(\mathcal{T}) \leq L \gamma_2(\mathcal{T}). \quad (38)$$

For the remainder of the proof, we take  $\mathcal{T} = D(K, \lambda \mathbf{x}) \cap \mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ .

Define

$$\overline{Z}(\mathbf{t}) = f_{\mathbf{x}}(\mathbf{t}) - E[f_{\mathbf{x}}(\mathbf{t})],$$

where  $f_{\mathbf{x}}(\mathbf{t})$  is as defined in (5) and

$$Z(\mathbf{t}) = \frac{1}{m} \sum_{i=1}^m \varepsilon_i y_i \langle \mathbf{a}_i, \mathbf{t} \rangle,$$

where  $\varepsilon_i, i = 1, \dots, m$  are Rademacher variables taking values uniformly in  $\{1, -1\}$ , independent of each other and of  $\{y_i, \mathbf{a}_i, i = 1, 2, \dots, m\}$ .

The majority of the proof of Lemma 3.2 is devoted to showing that

$$P \left[ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \geq C(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + u}{\sqrt{m}} \right] \leq e^{-u} \quad \text{for } u \geq 2, m \geq \omega(\mathcal{T})^2, \quad (39)$$

where  $C > 0$  is a constant.

Once (39) is justified, by the fact  $u \geq 2$ , we have

$$P \left[ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \geq C(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + 1}{\sqrt{m}} u \right] \leq e^{-u} \quad \text{for } u \geq 2, m \geq \omega(\mathcal{T})^2.$$

By Lemma A.4 with  $p = 1$ , and  $k = 1$ , we have

$$E \left[ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \right] \leq C(\tilde{c}_1 + 2)(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + 1}{\sqrt{m}}$$

where  $\tilde{c}_1$  is another constant. Thus, invoking the first bound in the symmetrization lemma, Lemma A.2,

$$\begin{aligned} E \left[ \sup_{\mathbf{t} \in \mathcal{T}} |\overline{Z}(\mathbf{t})| \right] &\leq 2E \left[ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \right] \leq 2C(\tilde{c}_1 + 2)(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + 1}{\sqrt{m}} \\ &\leq 2C(\tilde{c}_1 + 2)(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + u}{\sqrt{m}}, \quad \forall u \geq 2. \end{aligned}$$

We may then finish the proof of using the second bound in the symmetrization lemma with  $\beta = 2C(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + u}{\sqrt{m}}$  as well as (39). This gives

$$\begin{aligned} P \left[ \sup_{\mathbf{t} \in \mathcal{T}} |\overline{Z}(\mathbf{t})| \geq 2C(\tilde{c}_1 + 3)(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + u}{\sqrt{m}} \right] \\ \leq 4P \left[ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \geq C(\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\omega(\mathcal{T}) + u}{\sqrt{m}} \right] \leq 4e^{-u}, \end{aligned}$$

which is essentially Lemma 3.2 with constant  $C_0$  equal to  $2C(\tilde{c}_1 + 3)$ .

The rest of the section is devoted to the proof of (39). Pick  $\mathbf{t}_0 \in \mathcal{T}$  so that  $\{\mathbf{t}_0\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$  is an admissible sequence satisfying<sup>2</sup>

$$\sup_{\mathbf{t} \in \mathcal{T}} \sum_{l=0}^{\infty} 2^{l/2} d(\mathbf{t}, \pi_l(\mathbf{t})) \leq 2\gamma_2(\mathcal{T}), \quad (40)$$

where we recall  $\pi_l$  is the closest point map from  $\mathcal{T}$  to  $\mathcal{A}_l$ . Then, for any  $\mathbf{t} \in \mathcal{T}$ , we write  $Z(\mathbf{t}) - Z(\mathbf{t}_0)$  as a telescoping sum, i.e.

$$Z(\mathbf{t}) - Z(\mathbf{t}_0) = \sum_{l=1}^{\infty} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) = \sum_{l=1}^{\infty} \frac{1}{m} \sum_{i=1}^m \varepsilon_i y_i \langle \mathbf{a}_i, \pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t}) \rangle. \quad (41)$$

Note that this telescoping sum converges with probability 1 because the right hand side of (40) is finite.

Then, following ideas in [Di13], we fix an arbitrary positive integer  $p$  and let  $l_p := \lfloor \log_2 p \rfloor$ . Specializing (41) to the case  $t_0 = \pi_{l_p}(\mathbf{t})$  we obtain, with probability one, that

$$Z(\mathbf{t}) - Z(\pi_{l_p}(\mathbf{t})) = \sum_{l=l_p+1}^{\infty} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) = \sum_{l=l_p+1}^{\infty} \frac{1}{m} \sum_{i=1}^m \varepsilon_i y_i \langle \mathbf{a}_i, \pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t}) \rangle. \quad (42)$$

We split the outer indices of summation in (42) into the following two sets

$$I_{1,p} := \{l > l_p : 2^{l/2} \leq \sqrt{m}\} \quad \text{and} \quad I_{2,p} := \{l > l_p : 2^{l/2} > \sqrt{m}\}.$$

On the coarse scale  $I_{1,p}$ , we have the following lemma:

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<sup>2</sup>The constant 2 on the right hand side of the inequality is introduced to handle the case where the infimum within the definition of  $\gamma_2(T)$  is not achievable.

**Lemma 3.4** (Coarse scale chaining). *For all  $p \geq 1$ , the following inequality holds with probability at least  $1 - ce^{-pu/4}$ :*

$$\sup_{\mathbf{t} \in \mathcal{T}} \left| \sum_{l \in I_{1,p}} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) \right| \leq 4(\sqrt{2} + 1) \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \frac{u}{\sqrt{m}} \gamma_2(\mathcal{T}),$$

with some constant  $c > 0$  and all  $u \geq 2$ .

*Proof.* We assume  $I_{1,p}$  is non-empty, else the claim is trivial. By Proposition 3.1 and Definition 3.2, for any  $i \in \{1, 2, \dots, m\}$  we have

$$\|\varepsilon_i y_i \langle \mathbf{a}_i, \pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t}) \rangle\|_{\psi_1} \leq 2 \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2.$$

Thus, for each  $l \in I_{1,p}$ , applying Bernstein's inequality (Lemma A.6) to

$$Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) = \frac{1}{m} \sum_{i=1}^m \varepsilon_i y_i \langle \mathbf{a}_i, \pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t}) \rangle,$$

an average of independent subexponential random variables, we have that for all  $v \geq 1$ ,

$$P \left[ |Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t}))| \geq 2 \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \left( \frac{\sqrt{2v}}{\sqrt{m}} + \frac{v}{m} \right) \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2 \right] \leq 2e^{-v}.$$

Let  $v = 2^l u$  for some  $u \geq 2$ . Using that  $2^{l/2} \leq \sqrt{m}$  since  $l \in I_{1,p}$ , and that  $u \geq \sqrt{u}$ , we have

$$P \left[ |Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t}))| \geq 2 \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} (\sqrt{2} + 1) \frac{u}{\sqrt{m}} 2^{l/2} \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2 \right] \leq 2 \exp(-2^l u). \quad (43)$$

Define the event

$$\Omega_{l,\mathbf{t}} = \left\{ \omega : |Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t}))| \geq 2(\sqrt{2} + 1) \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \frac{u}{\sqrt{m}} 2^{l/2} \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2 \right\},$$

and let  $\Omega := \bigcup_{l \in I_{1,p}} \bigcup_{\mathbf{t} \in \mathcal{T}} \Omega_{l,\mathbf{t}}$ . As  $\mathcal{A}_{l-1} \subseteq \mathcal{A}_l$  and  $\mathcal{A}_l = \{\pi_l(\mathbf{t})\}_{\mathbf{t} \in \mathcal{T}}$  is finite, it follows that

$$\bigcup_{\mathbf{t} \in \mathcal{T}} \Omega_{l,\mathbf{t}} = \bigcup_{\mathbf{t} \in \mathcal{A}_l} \Omega_{l,\mathbf{t}}, \quad \forall l \in I_{1,p},$$

and, as  $u \geq 2$ , Lemma A.3 with  $k = 1$  may now be invoked to yield

$$P \left[ \bigcup_{l \in I_{1,p}, \mathbf{t} \in \mathcal{T}} \Omega_{l,\mathbf{t}} \right] \leq ce^{-pu/4},$$

for some  $c > 0$ . Thus, on the event  $\Omega^c$ , we have

$$\begin{aligned} \sup_{\mathbf{t} \in \mathcal{T}} \left| \sum_{l \in I_{1,p}} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) \right| &\leq \sup_{\mathbf{t} \in \mathcal{T}} \sum_{l \in I_{1,p}} |Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t}))| \\ &\leq \sup_{\mathbf{t} \in \mathcal{T}} 2(\sqrt{2} + 1) \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \frac{u}{\sqrt{m}} \sum_{l \in I_1} 2^{l/2} \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2 \\ &\leq \sup_{\mathbf{t} \in \mathcal{T}} 2(\sqrt{2} + 1) \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \frac{u}{\sqrt{m}} \sum_{l=1}^{\infty} 2^{l/2} \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2 \\ &\leq 4(\sqrt{2} + 1) \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \frac{u}{\sqrt{m}} \gamma_2(\mathcal{T}), \end{aligned}$$

where the last inequality follows from (40), finishing the proof.  $\square$

For the finer scale chaining, we will apply the following lemma whose proof is in the appendix.

**Lemma 3.5.** *For any  $\mathbf{t} \in \mathbb{R}^d$ ,  $u \geq 1$  and  $2^{l/2} > \sqrt{m}$ , we have*

$$P \left[ \left( \frac{1}{m} \sum_{i=1}^m \langle \mathbf{a}_i, \mathbf{t} \rangle^2 \right)^{1/2} \geq \sqrt{4 + 3\sqrt{2}} \|\mathbf{a}\|_{\psi_2} \sqrt{\frac{u}{m}} 2^{l/2} \|\mathbf{t}\|_2 \right] \leq 2 \exp(-2^l u).$$

**Lemma 3.6** (Finer scale chaining). *Let*

$$Y_m := \left| \frac{1}{m} \sum_{i=1}^m y_i^2 - E[y^2] \right|.$$

*Then for all  $p \geq 1$ , with probability at least  $1 - ce^{-pu/4}$*

$$\sup_{\mathbf{t} \in \mathcal{T}} \left| \sum_{l \in I_{2,p}} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) \right| \leq 2\sqrt{4 + 3\sqrt{2}} \left( Y_m^{1/2} + \sqrt{2} \|y\|_{\psi_2} \right) \|\mathbf{a}\|_{\psi_2} \sqrt{\frac{u}{m}} \gamma_2(\mathcal{T}),$$

*with some constant  $c > 0$  and  $u \geq 2$ .*

*Proof.* For any  $p \geq 1, l \in I_{2,p}$  and  $\mathbf{t} \in \mathcal{T}$ , by the Cauchy-Schwarz inequality,

$$\begin{aligned} |Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t}))| &= \left| \frac{1}{m} \sum_{i=1}^m \varepsilon_i y_i \langle \mathbf{a}_i, \pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t}) \rangle \right| \\ &\leq \left( \frac{1}{m} \sum_{i=1}^m y_i^2 \right)^{1/2} \cdot \left( \frac{1}{m} \sum_{i=1}^m |\langle \mathbf{a}_i, \pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t}) \rangle|^2 \right)^{1/2} \end{aligned}$$

Since  $y$  is subgaussian,  $E[y^2] \leq 2\|y\|_{\psi_2}^2$ . Thus,

$$\left( \frac{1}{m} \sum_{i=1}^m y_i^2 \right)^{1/2} = \left( \frac{1}{m} \sum_{i=1}^m y_i^2 - E[y^2] + E[y^2] \right)^{1/2} \leq Y_m^{1/2} + \sqrt{2} \|y\|_{\psi_2}.$$

Furthermore, by Lemma 3.5, for any  $l \in I_{2,p}$ , we have

$$P \left[ \left( \frac{1}{m} \sum_{i=1}^m |\langle \mathbf{a}_i, \pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t}) \rangle|^2 \right)^{1/2} \geq \sqrt{4 + 3\sqrt{2}} \|\mathbf{a}\|_{\psi_2} \sqrt{\frac{u}{m}} 2^{l/2} \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2 \right] \leq 2 \exp(-2^l u).$$

Thus, combining the above two inequalities,

$$\begin{aligned} P \left[ |Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t}))| \geq \sqrt{4 + 3\sqrt{2}} \left( Y_m^{1/2} + \sqrt{2} \|y\|_{\psi_2} \right) \|\mathbf{a}\|_{\psi_2} \sqrt{\frac{u}{m}} 2^{l/2} \|\pi_l(\mathbf{t}) - \pi_{l-1}(\mathbf{t})\|_2 \right] \\ \leq 2 \exp(-2^l u). \end{aligned}$$

The rest of the proof follows a standard chaining argument similar to the proof of Lemma 3.4 after (43) and is not repeated here for brevity.  $\square$

Now we are ready to prove Lemma 3.2. As shown at the start of Section 3.3, it only remains to show (39).

*Proof of (39).* First, for all  $p \geq 1$  and  $u \geq 2$ , by Lemma 3.6, with probability at least  $1 - ce^{-pu/4}$ ,

$$\begin{aligned} & \sup_{\mathbf{t} \in \mathcal{T}} \left| \sum_{l \in I_{2,p}} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) \right| \\ & \leq 2\sqrt{4 + 3\sqrt{2}Y_m^{1/2}} \|\mathbf{a}\|_{\psi_2} \sqrt{\frac{u}{m}} \gamma_2(\mathcal{T}) + 2\sqrt{8 + 6\sqrt{2}} \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \sqrt{\frac{u}{m}} \gamma_2(\mathcal{T}) \\ & \leq Y_m + (4 + 3\sqrt{2}) \|\mathbf{a}\|_{\psi_2}^2 \frac{u}{m} \gamma_2(\mathcal{T})^2 + 2\sqrt{8 + 6\sqrt{2}} \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \sqrt{\frac{u}{m}} \gamma_2(\mathcal{T}), \end{aligned}$$

where we applied the inequality  $2ab \leq a^2 + b^2$  on the first term. Then, combining with Lemma 3.4, we have with probability at least  $1 - ce^{-pu/4}$ ,

$$\begin{aligned} & \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t}) - Z(\pi_{l_p}(\mathbf{t}))| \\ & \leq \sup_{\mathbf{t} \in \mathcal{T}} \left| \sum_{l \in I_{1,p}} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) \right| + \sup_{\mathbf{t} \in \mathcal{T}} \left| \sum_{l \in I_{2,p}} Z(\pi_l(\mathbf{t})) - Z(\pi_{l-1}(\mathbf{t})) \right| \\ & \leq Y_m + (4 + 3\sqrt{2}) \|\mathbf{a}\|_{\psi_2}^2 \frac{u}{m} \gamma_2(\mathcal{T})^2 + 2\sqrt{8 + 6\sqrt{2}} \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \sqrt{\frac{u}{m}} \gamma_2(\mathcal{T}) \\ & \quad + 4(\sqrt{2} + 1) \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \frac{u}{\sqrt{m}} \gamma_2(\mathcal{T}) \\ & \leq Y_m + (4 + 3\sqrt{2}) \|\mathbf{a}\|_{\psi_2}^2 \frac{u}{m} \gamma_2(\mathcal{T})^2 + \left( \sqrt{8 + 6\sqrt{2}} + 2(\sqrt{2} + 1) \right) 2 \|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \frac{u}{\sqrt{m}} \gamma_2(\mathcal{T}). \end{aligned}$$

By the assumptions in (39) we have  $m \geq \omega(\mathcal{T})^2$ . Using inequality (38) on the relation between  $\omega(\mathcal{T})$  and  $\gamma_2(\mathcal{T})$  gives  $m \geq \gamma_2(\mathcal{T})^2/L^2$ . Thus,  $\gamma_2(\mathcal{T})^2/m \leq L\gamma_2(\mathcal{T})/\sqrt{m}$ , and the second term is bounded by

$$(4 + 3\sqrt{2})L \|\mathbf{a}\|_{\psi_2}^2 \frac{u}{\sqrt{m}} \gamma_2(\mathcal{T}).$$

For the last two terms we apply the bound  $2\|\mathbf{a}\|_{\psi_2} \|y\|_{\psi_2} \leq \|\mathbf{a}\|_{\psi_2}^2 + \|y\|_{\psi_2}^2$ . Thus, with probability at least  $1 - ce^{-pu/4}$ ,

$$\sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t}) - Z(\pi_{l_p}(\mathbf{t}))| \leq Y_m + C (\|\mathbf{a}\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{u\gamma_2(\mathcal{T})}{\sqrt{m}},$$

for the constant

$$C = 4L + 2 + (3L + 2)\sqrt{2} + \sqrt{8 + 6\sqrt{2}}.$$

By Proposition 3.3,  $\|\mathbf{a}\|_{\psi_2} \leq C\|a\|_{\psi_2}$  for some constant  $C$ . Thus, with probability at least  $1 - ce^{-pu/4}$ , for some constant  $C$  large enough,

$$\sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t}) - Z(\pi_{l_p}(\mathbf{t}))| \leq Y_m + C (\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{u\gamma_2(\mathcal{T})}{\sqrt{m}},$$

or, equivalently

$$\xi \leq C (\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{u\gamma_2(\mathcal{T})}{\sqrt{m}} \quad \text{where} \quad \xi = \max \left\{ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t}) - Z(\pi_{l_p}(\mathbf{t}))| - Y_m, 0 \right\}.$$

Invoking Lemma A.4 with  $k = 1$ , for all  $1 \leq p < \infty$

$$E[\xi^p]^{1/p} \leq C (\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\gamma_2(\mathcal{T})}{\sqrt{m}}.$$

Since

$$\begin{aligned} \xi &\geq \max \left\{ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t}) - Z(\pi_{l_p}(\mathbf{t}))|, 0 \right\} - Y_m = \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t}) - Z(\pi_{l_p}(\mathbf{t}))| - Y_m \\ &\geq \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| - \sup_{\mathbf{t} \in \mathcal{T}} |Z(\pi_{l_p}(\mathbf{t}))| - Y_m, \end{aligned}$$

and  $\xi$  and  $Y_m$  are both non-negative, by Minkowski's inequality it follows that

$$\begin{aligned} E \left[ \left( \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \right)^p \right]^{1/p} &\leq E \left[ \left( \xi + \sup_{\mathbf{t} \in \mathcal{T}} |Z(\pi_{l_p}(\mathbf{t}))| + Y_m \right)^p \right]^{1/p} \\ &\leq E[\xi^p]^{1/p} + E \left[ \left( \sup_{\mathbf{t} \in \mathcal{T}} |Z(\pi_{l_p}(\mathbf{t}))| \right)^p \right]^{1/p} + E[Y_m^p]^{1/p} \\ &\leq C (\|a\|_{\psi_2}^2 + \|y\|_{\psi_2}^2) \frac{\gamma_2(\mathcal{T})}{\sqrt{m}} + E \left[ \left( \sup_{\mathbf{t} \in \mathcal{T}} |Z(\pi_{l_p}(\mathbf{t}))| \right)^p \right]^{1/p} + E[Y_m^p]^{1/p}. \quad (44) \end{aligned}$$

For the second term, we have

$$E \left[ \left( \sup_{\mathbf{t} \in \mathcal{T}} |Z(\pi_{l_p}(\mathbf{t}))| \right)^p \right] \leq \sum_{\mathbf{t} \in \mathcal{A}_{l_p}} E[|Z(\mathbf{t})|^p] \leq |\mathcal{A}_{l_p}| \sup_{\mathbf{t} \in \mathcal{T}} E[|Z(\mathbf{t})|^p] \leq 2^p \sup_{\mathbf{t} \in \mathcal{T}} E[|Z(\mathbf{t})|^p],$$

where the first inequality follows from the fact that  $\pi_{l_p}(\cdot)$  can only take values in  $\mathcal{A}_{l_p}$ , and the last inequality follows from the fact that  $l_p = \lfloor \log_2 p \rfloor$ . On the other hand, applying Proposition 3.3, yielding that  $\|\mathbf{a}\|_{\psi_2} \leq C\|a\|_{\psi_2}$ , and Proposition 3.1, by a direct application of Bernstein's inequality (Lemma A.6) we have, for any fixed  $\mathbf{t} \in \mathcal{T}$ ,

$$P \left[ |Z(\mathbf{t})| \geq 2C\|y\|_{\psi_2}\|a\|_{\psi_2}(1 + \sqrt{2}) \frac{pu}{\sqrt{m}} \right] \leq 2e^{-pu}, \quad \text{whenever } pu \geq 0.$$

Hence, by Lemma A.4,

$$E[|Z(\mathbf{t})|^p]^{1/p} \leq \frac{C\|y\|_{\psi_2}\|a\|_{\psi_2}p}{\sqrt{m}},$$

for all  $t \in \mathcal{T}$  and some constant  $C > 0$ . Thus,

$$E \left[ \left( \sup_{\mathbf{t} \in \mathcal{T}} |Z(\pi_{l_p}(\mathbf{t}))| \right)^p \right]^{1/p} \leq \frac{2C\|y\|_{\psi_2}\|a\|_{\psi_2}p}{\sqrt{m}} \leq \frac{C(\|y\|_{\psi_2}^2 + \|a\|_{\psi_2}^2)p}{\sqrt{m}}. \quad (45)$$

Now consider  $E[Y_m^p]^{1/p}$ , the final term in (44), and recall  $Y_m = \frac{1}{m} \sum_{i=1}^m (y_i^2 - E[y_i^2])$ . Applying Proposition 3.1, we have

$$\|y_i^2 - E[y_i^2]\|_{\psi_1} \leq \|y_i^2\|_{\psi_1} + E[y_i^2] \leq 2\|y_i\|_{\psi_2}^2 + 2\|y_i\|_{\psi_2}^2 = 4\|y\|_{\psi_2}^2.$$

Thus, by Bernstein's inequality again,

$$Pr \left[ Y_m \geq 4(1 + \sqrt{2})\|y\|_{\psi_2}^2 \frac{pu}{\sqrt{m}} \right] \leq 2e^{-pu}, \quad \forall pu \geq 0.$$



By Lemma A.4 with  $k = 1$ , for all  $1 \leq p < \infty$ , we obtain,

$$E[Y_m^p]^{1/p} \leq \frac{C\|y\|_{\psi_2}^2 p}{\sqrt{m}} \leq \frac{C\left(\|y\|_{\psi_2}^2 + \|a\|_{\psi_2}^2\right) p}{\sqrt{m}}. \quad (46)$$

Combining (44), (45) and (46) gives

$$E\left[\left(\sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})|\right)^p\right]^{1/p} \leq \frac{C\left(\|y\|_{\psi_2}^2 + \|a\|_{\psi_2}^2\right)(\gamma_2(\mathcal{T}) + p)}{\sqrt{m}},$$

for some constant  $C > 0$ . Since this inequality holds for any  $p \geq 1$ , applying Lemma A.5 with  $k = 1$  yields

$$P\left[\sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \geq C(\|y\|_{\psi_2}^2 + \|a\|_{\psi_2}^2) \frac{\gamma_2(\mathcal{T}) + u}{\sqrt{m}}\right] \leq e^{-u}.$$

The proof is now completed by invoking Lemma 3.1, which gives  $\gamma_2(\mathcal{T}) \leq L\omega(\mathcal{T})$  for some constant  $L \geq 1$ .  $\square$

## Appendix A: Additional lemmas

The following lemma is one version of the contraction principle; for a proof see [LT91]:

**Lemma A.1.** *Let  $F : [0, \infty) \rightarrow [0, \infty)$  be convex and nondecreasing. Let  $\{\eta_i\}$  and  $\{\xi_i\}$  be two symmetric sequences of real valued random variables such that for some constant  $C \geq 1$  and every  $i$  and  $t > 0$ ,*

$$P[|\eta_i| > t] \leq C \cdot P[|\xi_i| > t].$$

*Then, for any finite sequence  $\{\mathbf{x}_i\}$  in a vector space with semi-norm  $\|\cdot\|$ ,*

$$E\left[F\left(\left\|\sum_i \eta_i \mathbf{x}_i\right\|\right)\right] \leq E\left[F\left(C \cdot \left\|\sum_i \xi_i \mathbf{x}_i\right\|\right)\right].$$

**Remark A.1.** *Though Lemma 4.6 of [LT91] states the contraction principle in a Banach space, we allow vector spaces under any semi-norm by following the proofs of Theorem 4.4 and Lemma 4.6 of [LT91].*

The following symmetrization lemma is exactly the same as Lemma 4.6 of [ALPV14]. Its proof is similar to Lemma 6.3 and Lemma 6.5 in [LT91].

**Lemma A.2** (Symmetrization). *Let*

$$\overline{Z}(\mathbf{t}) = f_{\mathbf{x}}(\mathbf{t}) - E[f_{\mathbf{x}}(\mathbf{t})] \quad \text{where} \quad f_{\mathbf{x}}(\mathbf{t}) = \frac{1}{m} \sum_{i=1}^m y_i \langle \mathbf{a}_i, \mathbf{t} \rangle,$$

*and*

$$Z(\mathbf{t}) = \frac{1}{m} \sum_{i=1}^m \varepsilon_i y_i \langle \mathbf{a}_i, \mathbf{t} \rangle,$$

*where  $\{\varepsilon_i : 1 \leq i \leq m\}$  is a collection of Rademacher random variables, each uniformly distributed over  $\{-1, 1\}$ , and independent of each other and of  $\{y_i, \mathbf{a}_i : 1 \leq i \leq m\}$ . Then for any measurable set  $\mathcal{T} \subset \mathbb{R}^d$ ,*

$$E\left[\sup_{\mathbf{t} \in \mathcal{T}} |\overline{Z}(\mathbf{t})|\right] \leq 2E\left[\sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})|\right],$$

and for any  $\beta > 0$

$$P \left[ \sup_{\mathbf{t} \in \mathcal{T}} |\overline{Z}(\mathbf{t})| \geq 2E \left[ \sup_{\mathbf{t} \in \mathcal{T}} |\overline{Z}(\mathbf{t})| \right] + \beta \right] \leq 4P \left[ \sup_{\mathbf{t} \in \mathcal{T}} |Z(\mathbf{t})| \geq \beta/2 \right].$$

**Lemma A.3** (Lemma A.4 of [Di13]). *Fix  $1 \leq p < \infty$ ,  $0 < k < \infty$ ,  $u \geq 2$  and  $l_p := \lfloor \log_2 p \rfloor$ . For every  $l > l_p$ , let  $J_l$  be a finite index set and  $\{\Omega_{l,i}\}_{i \in J_l}$  a collection of events satisfying*

$$P[\Omega_{l,i}] \leq 2 \exp(-2^l u^k), \quad \forall i \in J_l.$$

*If  $|J_l| \leq 2^{2^{l+1}}$ , then for an absolute constant  $c \leq 16$ ,*

$$P \left[ \bigcup_{l > l_p} \bigcup_{i \in J_l} \Omega_{l,i} \right] \leq c \exp(-pu^k/4).$$

**Lemma A.4** (Lemma A.5 of [Di13]). *Fix  $1 \leq p < \infty$  and  $0 < k < \infty$ . Let  $\beta \geq 0$  and suppose that  $\xi$  is a nonnegative random variable such that for some  $c, u_* > 0$ ,*

$$P[\xi > \beta u] \leq c \exp(-pu^k/4), \quad \forall u \geq u_*.$$

*Then for a constant  $\tilde{c}_k > 0$  depending only on  $k$ ,*

$$E[\xi^p]^{1/p} \leq \beta(\tilde{c}_k c + u_*).$$

**Lemma A.5** (Proposition 7.11 of [FR13]). *If  $X$  is a non-negative random variable satisfying*

$$E[X^p]^{1/p} \leq b + ap^{1/k} \quad \forall p \geq 1,$$

*for positive real numbers  $a$  and  $k$ , and  $b \geq 0$ , then, for any  $u \geq 1$ ,*

$$P \left[ X \geq e^{1/k}(b + au) \right] \leq \exp(-u^k/k).$$

Finally, for the following result see Theorem 2.10 of [BLM13].

**Lemma A.6** (Bernstein's inequality). *Let  $X_1, \dots, X_m$  be a sequence of independent, mean zero random variables. If there exist positive constants  $\sigma$  and  $D$  such that*

$$\frac{1}{m} \sum_{i=1}^m E[|X_i|^p] \leq \frac{p!}{2} \sigma^2 D^{p-2}, \quad p = 2, 3, \dots$$

*then for any  $u \geq 0$ ,*

$$P \left[ \left| \frac{1}{m} \sum_{i=1}^m X_i \right| \geq \frac{\sigma}{\sqrt{m}} \sqrt{2u} + \frac{D}{m} u \right] \leq 2 \exp(-u).$$

*In particular, if  $X_1, \dots, X_m$  are all subexponential random variables, then,  $\sigma$  and  $D$  can be chosen as  $\sigma = \frac{1}{m} \sum_{i=1}^m \|X_i\|_{\psi_1}$  and  $D = \max_i \|X_i\|_{\psi_1}$ .*

## Appendix B: Additional proofs

With  $g$  a standard normal variable, we begin by considering the solution  $f$  to (33), the special case of the Stein equation

$$f'(x) - xf(x) = h(x) - Eh(g), \quad (47)$$

with the specific choice of test function  $h(x) = |x|$ .

**Lemma B.1.** *The solution  $f$  of (33) satisfies  $\|f''\| = 1$ .*

*Proof.* In general, when  $f$  solves (47) for a given test function  $h(\cdot)$  then  $-f(-x)$  solves (47) for  $h(-\cdot)$ . As in the case at hand  $h(x) = |x|$ , for which  $h(-x) = h(x)$ , it suffices to show that  $0 \leq f(x) \leq 1$  for all  $x > 0$ , over which range (33) specializes to

$$f'(x) - xf(x) = x - \sqrt{\frac{2}{\pi}}. \quad (48)$$

Taking derivative on both sides yields

$$f''(x) - f(x) - xf'(x) = 1,$$

and combining the above two equalities gives

$$f''(x) = (1 + x^2)f(x) + x \left( x - \sqrt{\frac{2}{\pi}} \right) + 1. \quad (49)$$

On the other hand, solving (48) via integrating factors gives, for all  $x > 0$ ,

$$\begin{aligned} f(x) &= -e^{x^2/2} \int_x^\infty \left( z - \sqrt{\frac{2}{\pi}} \right) e^{-z^2/2} dz = -1 + 2e^{x^2/2} \int_x^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = \\ &= -1 + 2e^{x^2/2}(1 - \Phi(x)), \end{aligned} \quad (50)$$

where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal.

For any  $x > 0$ , by classical upper and lower tail bounds for  $\Phi(\cdot)$ , we have

$$\frac{x}{\sqrt{2\pi}(1+x^2)} \leq e^{x^2/2}(1 - \Phi(x)) \leq \min \left\{ \frac{1}{2}, \frac{1}{x\sqrt{2\pi}} \right\},$$

which in turn implies, using (49) and (50), that for all  $x > 0$

$$0 \leq f''(x) \leq \min \left\{ x \left( x - \sqrt{\frac{2}{\pi}} \right) + 1, \frac{1}{x} \sqrt{\frac{2}{\pi}} \right\}.$$

Handling the cases  $0 < x \leq \sqrt{2/\pi}$  and  $x > \sqrt{2/\pi}$  separately, we see  $0 \leq f''(x) \leq 1$  for all  $x > 0$ , as desired.  $\square$

*Proof of Proposition 3.1.* First of all, by definition  $\|XY\|_{\psi_1} = \sup_{p \geq 1} p^{-1} E[|XY|^p]^{1/p}$ . Applying  $2ab \leq a^2 + b^2$  and Minkowski's inequality, for any  $\epsilon > 0$ ,

$$E[|XY|^p]^{1/p} \leq E \left[ \left| \frac{X^2}{2\epsilon} + \frac{\epsilon Y^2}{2} \right|^p \right]^{1/p} \leq \frac{1}{2\epsilon} E[X^{2p}]^{1/p} + \frac{\epsilon}{2} E[Y^{2p}]^{1/p}.$$

Applying the definition of the  $\psi_1$  norm, this inequality implies

$$\|XY\|_{\psi_1} \leq \frac{1}{2\epsilon} \|X^2\|_{\psi_1} + \frac{\epsilon}{2} \|Y^2\|_{\psi_1}.$$

The term  $\|X^2\|_{\psi_1}$  can be bounded as follows,

$$\|X^2\|_{\psi_1} = \sup_{p \geq 1} \left( p^{-1/2} E[X^{2p}]^{1/2p} \right)^2 = \sqrt{2} \sup_{p \geq 1} \left( (2p)^{-1/2} E[X^{2p}]^{1/2p} \right)^2 \leq 2 \|X\|_{\psi_2}^2.$$

Thus,

$$\|XY\|_{\psi_1} \leq \frac{1}{\epsilon} \|X\|_{\psi_2}^2 + \epsilon \|Y\|_{\psi_2}^2.$$

Finally, choosing  $\epsilon = \|X\|_{\psi_2} / \|Y\|_{\psi_2}$  finishes the proof.  $\square$

*Proof of Proposition 3.2.* By definition, we have

$$\begin{aligned} \|\mathbf{a}\|_{\psi_2} &= \sup_{\mathbf{z} \in \mathbb{S}^{d-1}} \|\langle \mathbf{a}, \mathbf{z} \rangle\|_{\psi_2} \\ &= \sup_{\mathbf{z} \in \mathbb{S}^{d-1}} \sup_{p \geq 1} \frac{1}{p^{1/2}} E[|\langle \mathbf{a}, \mathbf{z} \rangle|^p]^{1/p} \\ &\geq \sup_{\mathbf{z} \in \mathbb{S}^{d-1}} \frac{1}{\sqrt{2}} E[\langle \mathbf{a}, \mathbf{z} \rangle^2]^{1/2} \\ &= \frac{1}{\sqrt{2}} \sup_{\mathbf{z} \in \mathbb{S}^{d-1}} \langle \Sigma \mathbf{z}, \mathbf{z} \rangle^{1/2} \\ &= \frac{1}{\sqrt{2}} \sigma_{\max}(\Sigma)^{1/2}, \end{aligned}$$

and squaring both sides finishes the proof.  $\square$

*Proof of Proposition 3.3.* Let  $g, g_i, i = 1, 2, \dots, n$  be independent standard Gaussian random variables. Then, see for instance Lemma 5.5 of [Ve10], as  $a$  is subgaussian, there exists a constant  $C \geq 0$  such that

$$P[|a_i| \geq t] \leq C \|a\|_{\psi_2} \cdot P[|g_i| \geq t].$$

Thus, by Lemma A.1, using the fact that  $F(x) = x^p$  is convex and nondecreasing in  $x > 0$  for  $p \geq 1$ , we have for any  $\mathbf{z} \in \mathbb{S}^{d-1}$ ,

$$\begin{aligned} \|\langle \mathbf{a}, \mathbf{z} \rangle\|_{\psi_2} &= \sup_{p \geq 1} p^{-1/2} E[|\langle \mathbf{a}, \mathbf{z} \rangle|^p]^{1/p} = \sup_{p \geq 1} p^{-1/2} E \left[ \left| \sum_{j=1}^n a_j z_j \right|^p \right]^{1/p} \\ &\leq \sup_{p \geq 1} p^{-1/2} E \left[ \left[ C \|a\|_{\psi_2} \cdot \sum_{j=1}^n g_j z_j \right]^p \right]^{1/p} = C \|a\|_{\psi_2} \sup_{p \geq 1} p^{-1/2} E[|g|^p]^{1/p} = C \|a\|_{\psi_2}. \end{aligned}$$

$\square$

*Proof of Lemma 3.5.* Since  $\langle \mathbf{a}_i, \mathbf{t} \rangle$  is subgaussian, it follows,  $\langle \mathbf{a}_i, \mathbf{t} \rangle^2$  is subexponential by Proposition 3.1. Note that  $E[\langle \mathbf{a}_i, \mathbf{t} \rangle^2] \leq \sigma_{\max}(\Sigma) \|\mathbf{t}\|_2^2 \leq \|\mathbf{a}\|_{\psi_2}^2 \|\mathbf{t}\|_2^2$  by Proposition 3.2. Then, by Remark 1.3 and Proposition 3.1

$$\|\langle \mathbf{a}_i, \mathbf{t} \rangle^2 - E[\langle \mathbf{a}_i, \mathbf{t} \rangle^2]\|_{\psi_1} \leq \|\langle \mathbf{a}_i, \mathbf{t} \rangle^2\|_{\psi_1} + \|\mathbf{a}\|_{\psi_2}^2 \|\mathbf{t}\|_2^2 \leq 3 \|\mathbf{a}\|_{\psi_2}^2 \|\mathbf{t}\|_2^2.$$

Apply Bernstein's inequality (Lemma A.6) gives,

$$P \left[ \left( \frac{1}{m} \sum_{i=1}^m \langle \mathbf{a}_i, \mathbf{t} \rangle^2 - E[\langle \mathbf{a}_i, \mathbf{t} \rangle^2] \right) \geq 3 \|\mathbf{a}\|_{\psi_2}^2 \left( \frac{\sqrt{2v}}{\sqrt{m}} + \frac{v}{m} \right) \|\mathbf{t}\|_2^2 \right] \leq 2e^{-v}.$$

We let  $v = 2^l u$  and apply the hypothesis  $2^{l/2} > \sqrt{m}$  and  $u \geq 1$  to obtain

$$P \left[ \left( \frac{1}{m} \sum_{i=1}^m \langle \mathbf{a}_i, \mathbf{t} \rangle^2 - E[\langle \mathbf{a}_i, \mathbf{t} \rangle^2] \right) \geq 3 (1 + \sqrt{2}) \|\mathbf{a}\|_{\psi_2}^2 \frac{2^l u}{m} \|\mathbf{t}\|_2^2 \right] \leq 2 \exp(-2^l u).$$

Thus, by  $2^{l/2} > \sqrt{m}$  and  $u \geq 1$  again,

$$P \left[ \left( \frac{1}{m} \sum_{i=1}^m \langle \mathbf{a}_i, \mathbf{t} \rangle^2 \right) \geq \left( 3 (1 + \sqrt{2}) + 1 \right) \frac{2^l u}{m} \|\mathbf{a}\|_{\psi_2}^2 \|\mathbf{t}\|_2^2 \right] \leq 2 \exp(-2^l u),$$

which yields the claim upon taking square roots on both sides of the first inequality.  $\square$

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